Optimal Dynamic Joint Pricing and Inventory Control for Multiplicative Demand with Fixed Order Costs

Yuyue Song • Saibal Ray • Tamer Boyaci
{yuyue.song;saibal.ray;tamer.boyaci}@mcgill.ca
Desautels Faculty of Management, McGill University, Montreal, Canada

Abstract

In this paper, we study the optimal dynamic decision-making for a retailer selling a single product, and facing multiplicative price-sensitive stochastic demand in each period. Any leftover inventory for a period is carried over and charged a holding cost at the end of the period. On the other hand, any unsatisfied demand can either be backordered or lost. Leftovers at the end of the selling horizon earn a non-negative salvage value. The purchasing cost for the retailer includes both variable and fixed components. We allow all cost and demand parameters to be non-stationary in our framework. The retailer’s objective is to maximize its discounted expected profit over the selling horizon by jointly deciding on the optimal pricing and inventory control policy for each period. Under some mild conditions, which are common in the literature, we characterize the structure of the optimal policy for the retailer for both finite and infinite selling horizons.

Key Words: Joint inventory control and pricing, Multiplicative price-sensitive stochastic demand, Fixed order cost, Lost sales, Backordering.
1 Introduction

Our focus in this paper is on the optimal dynamic decision-making problem for a retailer dealing with a single product. We particularly address the two most strategic decisions - those of setting prices and planning for the amount of inventory to procure (Stern and El-Ansary 1992). The issue of dynamic inventory control has been of interest for operations management (OM) researchers for quite some time, and extensive results are available in this area (Porteus 2002). These models normally assume the retail price per unit to be constant over the planning horizon. However, recent advances in information technology make it relatively easy (and cheap) to change prices dynamically, especially in online environments. A number of retailers (e.g., Dell, Boise Cascade, Amazon.com) are already using dynamic pricing as a competitive strategy, and some of them (e.g., Dell) are even altering both pricing and inventory decisions regularly in order to increase their profits (Chen and Simchi-Levi 2004b). This has prompted an interest from academicians in determining the optimal dynamic policy jointly in terms of these two decisions (Elmaghraby and Keskinocak 2003).

The usual framework for such research is a periodic review setting with the end customer demand in each period being both stochastic and price-sensitive. We also adopt such a setting, but assume specifically that the demand in each period can be expressed as the product of a price-sensitive deterministic function and a non-price-sensitive positive random variable (usually termed as multiplicative demand). At each review epoch, the retailer needs to decide on the replenishment quantity from the manufacturer (which is received instantenously) and the price to charge to the end customers for that period. Obviously, there are considerable interactions between the two decisions. For example, a high price for a particular period would most probably result in low demand, and substantial leftover inventory at the end of the period. So the replenishment quantity for the next period would be low (maybe even zero). Conversely, a large quantity of unsold inventory at the end of a period might induce a small order and/or a low price in the following period.

We also assume the following for our model framework. The retailer’s replenishment cost involves two elements - a variable cost per unit and a fixed order cost for any positive purchase quantity. As far as unmet (excess) demands due to insufficient inventory are concerned, our focus is on the scenario where such demands are lost (resulting in lost revenue), although we also briefly discuss backordering of excess demands (and consequent backordering cost). Any leftover inventory from a period, on the other hand, is carried forward to satisfy future demand, and a holding cost is charged proportional to the inventory level at the end of the period. The leftovers at the end of the selling horizon are salvaged at a nonnegative per unit value. Note that all demand and cost
parameters can be non-stationary. The aim of this paper is to develop and analyze a dynamic joint pricing and inventory control policy that would maximize the retailer’s total discounted expected profit over the selling horizon, which can either be finite or infinite.

1.1 Literature Review

Dynamic analysis of optimal joint pricing and inventory control problem has recently become popular in the OM literature. Our objective here is not to provide a comprehensive review, but rather to position our work with respect to this literature. Most of the related literature can be categorized based on two primary characteristics - whether the excess demand in each period is backordered or lost, and whether the price-sensitive stochastic demand is of the multiplicative or of the additive form (i.e., price-sensitive deterministic function plus a non-price-sensitive positive random variable). A classification of the relevant research based on these characteristics is provided in Tables 1 and 2. Table 3 provides a summary and highlights the open research area.

When the demand form is additive, it is well-known that an \((s, S, p)\) policy is optimal. This means that at the beginning of each period an \((s, S)\) policy is optimal for replenishment, and the optimal price \(p\) depends on the post-replenishment inventory level. This result has been established for backordering and lost sales models by Chen and Simchi-Levi (2004 a, b) and Chen et al. (2005), respectively. Using an alternative approach, Huh and Janakiraman (2005) also proves optimality of the \((s, S, p)\) policy for both cases under quite general conditions (details below).

The optimal policy for multiplicative demand is more involved. If excess demands are backordered, Chen and Simchi-Levi (2004 a, b) has shown that an \((s, S, p)\) policy is not necessarily optimal. This is because the total profit function, at least for a certain range of initial stock levels, is not necessarily well-behaved. Using an elegant concept called symmetric \(K\)-convexity, they have shown that the optimal policy is of the \((s, S, A, p)\) form for the finite horizon problem. Under this policy, if the starting inventory of a period is:

i) Less than \(s\) or belongs to set \(A (A \subseteq [s, (s+S)/2])\), then the optimal order-up-to level is \(S\). Note that \(A\) is the set of all inventory levels within \([s, (s+S)/2]\) for which ordering is optimal. As such, this set is not known a-priori. It needs to be determined via some search, and can well be empty.

ii) Greater than \((s+S)/2\) or belongs to set \([s, (s+S)/2] \cap A\), no order is placed.

The optimal \(p\) depends on the post-replenishment inventory level. To the best of our knowledge, the structure of the optimal policy when any excess demand is lost remains an open question.
According to Polatoglu and Sahin (2000), multiple optimal order-up-to levels might exist in this case (also refer to Chen and Simchi-Levi, 2004a, pg 892).

<table>
<thead>
<tr>
<th></th>
<th>Demand Form</th>
<th>Riskless/Random</th>
<th>Fixed C.</th>
<th>Optimal Policy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thomas (1975)</td>
<td>general</td>
<td>general/general</td>
<td>positive</td>
<td>not clear</td>
</tr>
<tr>
<td>Federgruen and Heching (1999)</td>
<td>general</td>
<td>general/general</td>
<td>zero</td>
<td>base-stock</td>
</tr>
<tr>
<td>Chen and Simchi-Levi (2004a)</td>
<td>additive</td>
<td>general/general</td>
<td>positive</td>
<td>(s, S, p)</td>
</tr>
<tr>
<td>Chen and Simchi-Levi (2004a)</td>
<td>multiplicative</td>
<td>general/general</td>
<td>positive</td>
<td>(s, S, A, p)</td>
</tr>
</tbody>
</table>

Table 1: **Dynamic Joint Inventory Control and Pricing Models With Backordering**

<table>
<thead>
<tr>
<th></th>
<th>Demand Form</th>
<th>Riskless/Random</th>
<th>Fixed C.</th>
<th>Optimal Policy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zabel (1972)</td>
<td>additive</td>
<td>general/uniform</td>
<td>zero</td>
<td>partial base-stock</td>
</tr>
<tr>
<td>Polatoglu and Sahin (2000)</td>
<td>additive</td>
<td>general/zero</td>
<td>positive</td>
<td>(s, S, p)</td>
</tr>
<tr>
<td>Polatoglu and Sahin (2000)</td>
<td>general</td>
<td>general/general</td>
<td>positive</td>
<td>not clear</td>
</tr>
<tr>
<td>Chen et al. (2005)</td>
<td>additive</td>
<td>general/general</td>
<td>positive</td>
<td>(s, S, p)</td>
</tr>
</tbody>
</table>

Table 2: **Dynamic Joint Inventory Control and Pricing Models With Lost Sales**

<table>
<thead>
<tr>
<th></th>
<th>Additive demand</th>
<th>Multiplicative Demand</th>
</tr>
</thead>
<tbody>
<tr>
<td>Backordering</td>
<td>(s, S, p) policy optimal</td>
<td>(s, S, A, p) policy optimal</td>
</tr>
<tr>
<td>Lost Sales</td>
<td>(s, S, p) policy optimal</td>
<td>unknown</td>
</tr>
</tbody>
</table>

Table 3: **Overall Summary**

Motivated by above, we focus on the multiplicative demand form with a fixed cost associated with any replenishment. Our primary contribution is to establish the optimality of the (s, S, A, p) policy for a finite selling horizon when all excess demands are lost (i.e., open question). We prove this result by using an approach different than the one adopted by Chen and Simchi-Levi (2004a) for the backordering case. However, our technique readily applies to the backordering case, re-establishing their result regarding the optimality of the (s, S, A, p) policy. We only discuss this briefly in our paper. Note that due to the structure of the (s, S, A, P) policy, the values of the optimal policy parameters may well be different in the two papers for the backordering scenario. Our approach provides a simpler way of computing these values and new insights on the behavior of the optimal profit function. In addition to above, we show that for the special case with zero fixed order costs, the optimal policy naturally reduces to a base-stock policy, whereas for the stationary infinite horizon problem, an (s, S, p) policy is optimal (for both lost sales and backordering).
Our optimality proofs borrow some concepts and results from Huh and Janakiraman (2005). Hence, we briefly discuss that paper before presenting our results. Huh and Janakiraman first identify a property ("unifying condition") related to the single period expected profit function. Under two key assumptions on this "unifying condition", and another "monotonicity property", they prove that the \((s, S, p)\) policy is optimal under very general conditions. However, the "unifying condition" and the "monotone property" need to be studied for specific scenarios in order to verify their validity. This verification process is non-trivial. Huh and Janakiraman (2005) show that the "unifying condition" holds for: i) backordering models for both additive and multiplicative demand, and ii) lost sales models, but only for additive demand. They also show that the "monotone property" holds true for models with additive demand form. In effect, the two results establish the optimality of the \((s, S, p)\) policy for both lost sales and backordering scenarios, under additive demand. However, no such conclusion can be drawn for the lost sales model under multiplicative demand. In this paper we first prove a key property, which implies that their "unifying condition" is valid for this case. Subsequently, we establish certain new properties of the expected profit function for lost sales and backordering models. These results, along with the "unifying condition" for the backordering case already verified by Huh and Janakiraman (2005), enable us to establish the optimality results.

The remainder of this paper is organized as follows. In §2, we present the details of our model framework. §3 develops the optimal policy results for the lost sales model, and include a short discussion about the validity of these results for the backordering scenario. We end the paper with our concluding remarks in §4.

2 The General Model Framework

Suppose that the periodic review (finite) selling horizon has \(T\) periods indexed forwardly by period index \(t, 1 \leq t \leq T\). If the retail price charged in period \(t\) is \(p\), we assume the multiplicative price-sensitive demand in that period to be \(D_t(p)\epsilon_t\), where \(D_t(p)\) is a strictly decreasing\(^1\) deterministic function and \(\epsilon_t\) is a random variable, defined and positive on \((A_t, B_t), 0 \leq A_t < B_t < \infty\), with \(\mu_t = E[\epsilon_t]\). Let \(f_t(u)\) and \(F_t(u)\) be the density and the distribution functions of \(\epsilon_t\), respectively. Note that the demands for different periods need to be independent in our setting, but not necessarily be identically distributed. We assume that \(\lim_{u \rightarrow A_t^+} f_t(u) > 0\) if \(A_t = 0\) and \(f_t(u) = 0\) for any \(u \in [0, A_t]\) and \(B_t, +\infty\). We also assume \(\epsilon_t\) to have an increasing generalized failure rate (IGFR):

\(^1\)Throughout the paper we use increasing and decreasing in the weak sense, unless otherwise stated.
Assumption 1 \( r_t(u) = \frac{uf_t(u)}{-F_t(u)} \) is increasing in \( u \) on \((A_t, B_t)\) for any \( t(1 \leq t \leq T) \).

IGFR is a mild requirement satisfied by most distributions used in the OM literature like Uniform, Gamma with shape parameter \( \geq 1 \), Beta with both parameters \( \geq 1 \), Weibull with shape parameter \( \geq 1 \), Normal, Exponential, Left-truncated (at 0) Normal, and Left-truncated (at 0) Logistic (refer to Lariviere, 2004, for more details).

Let us define \( \Lambda_t(z) = \int_0^z(z-u)f_t(u)du \) and \( \Theta_t(z) = \int_z^{+\infty}(u-z)f_t(u)du \) for any \( z \in [0, +\infty) \), as in Petruzzi and Dada (1999). Notice that

\[
\begin{align*}
\int_0^z\mu_t\ - \Theta_t(z) &= \int_0^zuf_t(u)du \\
\mu_t\ - \Theta_t(z) &= \int_0^zu f_t(u)du + z[1 - F_t(z)].
\end{align*}
\]

We can then show that \( \epsilon_t \) exhibits the following property (refer to Song et al. 2005 for proof):

Lemma 1 Let \( V_t(z) = \frac{\mu_t - \Theta_t(z)}{\int_0^zu f_t(u)du} \) for any \( z \in (A_t, B_t) \). Then, \( V_t(z) \) is strictly decreasing on \((A_t, B_t)\), \( \lim_{z \to A_t} V_t(z) = +\infty \), and \( \lim_{z \to B_t} V_t(z) = 1 \).

The retailer’s problem can be described as follows. Given an initial stock level \( x(\geq 0) \) before ordering, at the beginning of period \( t \), the retailer needs to decide on the order-up-to inventory level \( y(\geq x) \) or equivalently the order quantity \((y-x)\), and the retail price \( p \), before any demand is realized. The objective is to maximize the total discounted expected profit from period \( t \) until the end of the planning horizon \( T \). Note that for an order of \((y-x)\) from the manufacturer, the retailer’s replenishment cost is given by

\[
K_t\delta(y-x) + w_t(y-x),
\]

where \( w_t(\geq 0) \) is the per unit purchasing cost, \( K_t(\geq 0) \) is the fixed order cost, and \( \delta(y-x) = 1 \) if \( y > x \) and 0 otherwise. Once the order is placed, it is received immediately by the retailer, after which the demand in period \( t \) is realized. The demand depends on the price \( p \) and realization of \( \epsilon_t \).

Let \( \pi_t(x) \) be the optimal discounted expected total profit from period \( t \) until the end of the planning horizon \( T \), when the starting inventory level is \( x \). Define \( g_t(y, p) \) as the expected profit in period \( t \) when \( x = 0 \) and \( K_t = 0 \). If \( 0 < \alpha \leq 1 \) denote the (given) discount factor, then the retailer’s maximization problem can be represented by the following dynamic programming (DP) formulation:

\[
\pi_t(x) = \max_{\{p > 0, y \geq x\}} \{ -K_t\delta(y-x) + g_t(y, p) + \alpha \int_0^{+\infty} \pi_{t+1}(y-D_t(p)u)f_t(u)du \} + w_t x, \tag{2}
\]

where \( \pi_{T+1}(x) = v^+x \) for any \( x \geq 0 \) and \( \pi_{T+1}(x) = v^-x \) otherwise. Note that the terminal salvage value/cost \((v^+/v^-)\) would take different values depending on whether excess demands are backordered or lost.
3 Model Analysis

In this section (except §3.3), we focus on analyzing the lost sales case. In §3.3 we confirm that our analysis procedure and results remain valid also for backordering of excess demands.

In the lost sales scenario, we restrict ourselves to iso-elastic demand function \( D_t(p) = p^{-k_t} \), where \( k_t > 1 \) is the price-elasticity. We note that for multiplicative demand, this is the usual assumption (see Petruzzi and Dada, 1999, 2002). Furthermore, because of lost sales, \( v^+ = v \geq 0 \) and \( v^- = 0 \). We do not assume an explicit penalty cost for lost demand (other than the foregone revenue), but charge a linear holding cost (at rate \( h_t \)) when leftover inventory \((y - D_t(p)u)\), if any, is carried to the next period. The exact expression of \( g_t(y, p) \) will then be:

\[
g_t(y, p) = pD_t(p)[\mu_t - \Theta_t(\frac{y}{D_t(p)})] - h_t\Lambda_t(\frac{y}{D_t(p)})D_t(p) - w_t y.
\]

In the above expression, the first term denotes the total expected revenue, the second term represents the expected holding cost, and the last term is the (variable) replenishment cost. Note that since any excess demand is lost, we define \( \pi_t(x) = \pi_t(0) \) if \( x < 0 \).

Define \( w_{T+1} = v \). Also, for any \( t(1 \leq t \leq T + 1) \), let \( \Pi_t(x) = \pi_t(x) - w_t x \) for any \( x \geq 0 \) and \( \Pi_t(x) = \Pi_t(0) \) otherwise. The DP formulation in (2) can be equivalently stated as:

\[
\Pi_t(x) = \max_{\{p > 0, y \geq x\}} \{-K_t \delta(y - x) + \Gamma_t(y, p) + \alpha \int_0^{+\infty} \Pi_{t+1}(y - D_t(p)u)f_t(u)du\},
\]

for any \( t(1 \leq t \leq T) \), where \( \Pi_{T+1}(x) = 0 \) for any \( x \) and

\[
\Gamma_t(y, p) = pD_t(p)[\mu_t - \Theta_t(\frac{y}{D_t(p)})] + (\alpha w_{t+1} - h_t)\Lambda_t(\frac{y}{D_t(p)})D_t(p) - w_t y.
\]

Note that \( \Gamma_t(y, p) \) is the total expected profit function for a price-setting newsvendor with a salvage value of \((\alpha w_{t+1} - h_t)\) (say, \( v_t \)) and zero initial stock. We assume this salvage value to be nonnegative and less than \( w_t \) throughout this paper. This basically implies that it is cheaper to procure a unit than to carry it over from the previous period, eliminating the “speculative” motive for holding inventory.

In the sequel, we analyze the behavior of the single period profit function, and based on those results, characterize the jointly optimal ordering and pricing policy for the multi-period version.

3.1 Single Period Model with Lost Sales

The expected single-period profit of the retailer with an initial inventory level \( x \) can be written as

\[
-K_t \delta(y - x) + \Gamma_t(y, p) + w_t x,
\]
where $\Gamma_t(y, p)$ is given by (4). Let us first characterize the behavior of $\Gamma_t(y, p)$.\(^2\)

**Theorem 1** Given any order-up-to inventory level $y(\geq x)$ for a period (say, $t$), there exists a unique $P_t(y)$, solution of $\frac{\partial \Gamma_t(y, p)}{\partial y} = 0$, such that $\Gamma_t(y, p)$ is maximized. Moreover, $\Gamma_t(y, P_t(y))$ is concave in $y$. Furthermore, the following results hold true:

1. There is a unique maximizer of $\Gamma_t(y, P_t(y))$ (say, $S_t$), and let $s_t(\leq S_t)$ be the maximal inventory level such that $\Gamma_t(s_t, P_t(s_t)) \leq \Gamma_t(S_t, P_t(S_t)) - K_t$. If there is no such $s_t$, define $s_t = 0$. Then, an $(s_t, S_t, P_t)$ policy is optimal for the retailer. That is, if $x \leq s_t$, the optimal order-up-to level is $S_t$ and the optimal price is $P_t(S_t)$; if $x > s_t$, then it is optimal not to order anything and set the price as $P_t(x)$.

2. Let $\Gamma_t(x)$ be the retailer’s optimal single period expected profit: $\Gamma_t(x) = \Gamma_t(S_t, P_t(S_t)) - K_t + w_t x$ for any $x \leq s_t$, and $\Gamma_t(x) = \Gamma_t(x, P_t(x)) + w_t x$ for any $x > s_t$. Also, $\Gamma_t'(x) \geq 0, \Gamma_t''(x) \leq 0$, and $\Gamma_t'(x) + k_t x \Gamma_t''(x) \geq 0$.

3. Let $Z_t(y) = \frac{y}{\Gamma_t'(P_t(y))}$ for any $y > 0$. $Z_t(y)$ is increasing in $y$. Hence, the leftover at the end of the period, $\max\{y - D_t(P_t(y))u, 0\}$, is also increasing in $y$ for any realization $u$ of $\epsilon_t$.

4. If $r_t(z)V_t(z) \geq 1$, then the optimal price $P_t(S_t)$ is decreasing and the optimal order-up-to level $S_t$ is increasing in the price elasticity $k_t$ of the product.$^3$

\(^2\)Proofs for Lemmas and Theorems which are not presented in the text, are provided in the Appendix.

\(^3\)Note that $r_t(z)V_t(z) \geq 1$ is related only to the random part of the single-period demand, and can be easily verified.

### 3.2 Multi-Period Model with Lost Sales

The above properties of $\Gamma_t(y, p)$ and the definitions of $s_t$ and $S_t$ enable us to characterize the optimal policy for the general multi-period model. Recall that for any given initial inventory $x$ at the beginning of period $t$, the retailer maximizes the expected total profit given by

$$-K_t \delta(y - x) + \Gamma_t(y, p) + \alpha \int_0^{+\infty} \Pi_{t+1}(y - D_t(p)u)f_t(u)du,$$

by selecting the order-up-to inventory level $y$ and the retail price $p$.

We approach this optimization problem sequentially. We first determine the optimal price for a given order-up-to level $y$. We then analyze the resulting one-variable problem in terms of $y$. To this end, define

$$H_t(y) = \max_p \{ \Gamma_t(y, p) + \alpha \int_0^{+\infty} \Pi_{t+1}(y - D_t(u)f_t(u)du) \}.$$
Let \( p_t(y_t) \) denote the optimal price for a given \( y_t \). If there are multiple optimal prices, we define \( p_t(y_t) \) as the smallest of them all. Let us also denote the maximizer of \( H_t(y) \) as \( y_t^* \). We now present a crucial property of \( \Gamma_t(y, p) \), which will help us to analyze \( H_t(y) \). Note that for any given joint decision pair \((y, p)\) (or a point on the two-dimensional plane), we define the leftover of the point \((y, p)\) as \( \max\{y - D_t(p)e_t, 0\} \).

**Lemma 2** For any given two order-up-to levels \( y^1, y^2 \) such that \( S_t \leq y^1 < y^2 \) and a given retail price \( p^2 \), there exists a retail price \( p^1 \) such that \( \Gamma_t(y^1, p^1) \geq \Gamma_t(y^2, p^2) \) and the leftover of the point \((y^2, p^2)\) is larger than or equal to the leftover of the point \((y^1, p^1)\). \( \square \)

The above result is slightly stronger and hence implies the “unifying condition” of Huh and Janakiraman (2005) for multiplicative demand with lost sales. It means that: i) nearer the inventory level at the beginning of period \( t \) to \( S_t \), better it is for the retailer from the viewpoint of expected profit; and ii) if we start with a higher inventory level at the beginning of the period, this will result in higher leftovers at the end of the period. Lemma 2 results in the following optimal replenishment strategy when the initial stock level is “high” (i.e., \( x \geq S_t \)).

**Lemma 3** Suppose \( K_l \geq \alpha K_{l+1} \) for any \( l \leq T \) and \( x \geq S_t \). Then it is optimal for the retailer to order nothing at the beginning of period \( t \).

**Proof:** Follows from Proposition 3.2 in Huh and Janakiraman (2005) and Lemma 2. \( \square \)

Note that \( K_l \geq \alpha K_{l+1} \) is a standard assumption in the related literature (e.g., Huh and Janakiraman 2005; Chen and Simchi-Levi 2004a). This is also realistic, since we would expect the fixed cost to reduce over time with experience and learning. However, we are still not able to make any judgment about the optimal policy when the initial stock level \( x \in [0, S_t] \). The following lemma facilitates answering this question.

**Lemma 4** Let \( \rho_t = \min\{t \leq l \leq T\} s_l \). Suppose \( S_l \leq S_{l+1} \) and \( K_l \geq \alpha K_{l+1} \) for any \( l \leq T \). Then the following are true:

1. \( H_t(y) \) is strictly increasing and \( H_t(y) \leq H_t(S_t) - K_t \) for any \( y \in [0, \rho_t] \).
2. \( H_t(y) > H_t(\rho_t) \) and \( H_t(y) \leq K_t + H_t(\bar{y}) \) for any \( \rho_t < y \leq \bar{y} \leq S_t \).

**Proof:** By Theorem 1, the proposition holds true for \( t = T \). Now suppose that it holds true for period \( t + 1(\leq T) \). In the following we will show it also holds true for period \( t \).
For period $t + 1$, we have $H_{t+1}(y) \leq H_{t+1}(S_{t+1}) - K_{t+1}$ and $H_{t+1}(y)$ is strictly increasing on $[0, \rho_{t+1}]$. This means that for any $x \in [0, \rho_{t+1}]$, it is always optimal to order up to $y_{t+1}^*$ at the beginning of period $t + 1$. Hence, $\Pi_{t+1}(x)$ is constant (say, $V_{t+1}$) on $[0, \rho_{t+1}]$. Based on the expression of $H_t(y)$, for any $y \in [0, \rho_t]$ we get $H_t(y) = V_{t+1} + \Gamma_t(y, P_t(y))$. Thus, $H_t(y)$ is strictly increasing on $[0, \rho_t]$.

Based on the relationship between $y_{t+1}^*$ and $S_{t+1}$, there are two possibilities. First, we consider the possibility of $y_{t+1}^* \geq S_{t+1}$. Then it is obvious that for any $\rho_t < y \leq \overline{y} \leq S_{t+1}$, we have

$$\Pi_{t+1}(y) \geq \Pi_{t+1}(\rho_t) = \Pi_{t+1}(\rho_t) \text{ and } |\Pi_{t+1}(y) - \Pi_{t+1}(\overline{y})| \leq K_{t+1}. \quad (5)$$

Secondly, we consider the possibility of $y_{t+1}^* \in (\rho_{t+1}, S_{t+1})$. If $y \in (\rho_t, y_{t+1}^*)$, clearly $\Pi_{t+1}(y) \geq \Pi_{t+1}(\rho_t)$. If $y \in (y_{t+1}^*, S_{t+1})$, as $H_{t+1}(y) \geq H_{t+1}(y_{t+1}^*) - K_{t+1} = \Pi_{t+1}(\rho_t)$ by our assumption that all the results in the lemma hold true for period $t + 1$, we get $\Pi_{t+1}(y) \geq \Pi_{t+1}(\rho_t)$. Thus, the first inequality in (5) also hold true under this second possibility. Similarly, we can argue that the second inequality in (5) also hold true under this second possibility.

Recall that $\Pi_{t+1}(y) = V_{t+1}$, a constant for any $y \in [0, \rho_t]$. We define $\hat{H}_t(y) = \Gamma_t(y, P_t(y)) + \alpha \int_0^{\infty} \Pi_{t+1}(y - D_t(P_t(y))u)f_t(u)du$. As $\Pi_{t+1}(y) \geq \Pi_{t+1}(\rho_t) = V_{t+1}$ by the first inequality in (5), we get

$$H_t(S_t) \geq \hat{H}_t(S_t) \geq \Gamma_t(S_t, P_t(S_t)) + \alpha V_{t+1} \geq \Gamma_t(\rho_t, P_t(\rho_t)) + K_t + \alpha V_{t+1}.$$ 

The last inequality is obtained from the definition of $s_t$ and the relation $\rho_t \leq s_t$. Hence, $H_t(S_t) \geq \hat{H}_t(\rho_t) + K_t = H_t(\rho_t) + K_t$. Thus, we obtain $H_t(y) \leq H_t(\rho_t) \leq H_t(S_t) - K_t$ for any $y \in [0, \rho_t]$ and we complete the proof of part (1).

For any $y \in (\rho_t, S_t]$, again by the first inequality in (5), we have

$$H_t(y) \geq \hat{H}_t(y) \geq \Gamma_t(y, P_t(y)) + \alpha V_{t+1} > \Gamma_t(\rho_t, P_t(\rho_t)) + \alpha V_{t+1} = \hat{H}_t(\rho_t) = H_t(\rho_t).$$

This proves the first result in part (2). Now we show the second inequality in part (2) is true. For any given $\rho_t < y \leq \overline{y} \leq S_t$, note that

$$H_t(y) = \Gamma_t(y, p_t(y)) + \alpha \int_0^{\infty} \Pi_{t+1}(y - D_t(p_t(y))u)f_t(u)du$$

$$\leq \Gamma_t(y, P_t(y)) + \alpha \int_0^{\infty} \Pi_{t+1}(y - D_t(p_t(y))u)f_t(u)du$$

$$\leq \Gamma_t(\overline{y}, P_t(\overline{y})) + \alpha \int_0^{\infty} \Pi_{t+1}(y - D_t(p_t(y))u)f_t(u)du$$

$$= \hat{H}_t(\overline{y}) + \alpha \int_0^{\infty} [\Pi_{t+1}(y - D_t(p_t(y))u) - \Pi_{t+1}(\overline{y} - D_t(P_t(\overline{y}))u)]f_t(u)du.$$
Combining the above and the second inequality in (5), we have $H_t(y) \leq \hat{H}_t(y) + \alpha K_{t+1} \leq H_t(y) + K_t$.

This completes the whole proof. \hfill \Box

Lemmas 3 and 4 jointly characterize the overall optimal dynamic decision policy for the multi-period model.

**Theorem 2** Suppose $K_l \geq \alpha K_{l+1}$ and $S_l \leq S_{l+1}$ for any $t$ ($t \leq l < T$). Then, at the beginning of period $t$, if the initial stock level $x \leq \rho_t$ or if $x \in (\rho_t, S_t)$ and it is optimal to order, then the retailer should order-up-to $y_t^*$ and set the price as $p_t(y_t^*)$. For any other initial stock level, the retailer’s optimal policy is to order nothing and charge the price $p_t(x)$.

**Proof:** We only need to show that for any initial stock level $x$, if it is optimal to order, then the optimal order-up-to inventory level is $y_t^*$. We prove this claim by a contradiction argument.

Suppose that the above claim is not true. There should exist four values: $x_1 < x_1^* < x_2 < x_2^*$ such that at $x_1$ it is optimal to order up to $x_1^*$ and at $x_2$ it is optimal to order up to $x_2^*$. As a consequence of this, we have $x_1 < x_2 \leq S_t$ by Lemma 3 and $H_t(x_1^*) > K_t + H_t(x_2)$. But, this is not possible by Lemma 4. Hence, the claim does hold true and we complete the proof. \hfill \Box

As Theorem 2 establishes, the optimal price to charge in each period is based on the post-replenishment inventory level. The optimal replenishment policy, on the other hand, is clear when the starting inventory level is either low ($x \leq \rho_t$) or high ($x \geq S_t$). The complication arises in the intermediate range ($\rho_t < x < S_t$) where it is not possible to ascertain the exact behavior of the profit function $H(y)$. Polatoglu and Sahin (2000) indicate that multiple order-up-to levels might exist (for general $D_t(p)$ functions). Nevertheless, we are able to show that for iso-elastic demand function, there is a unique order-up-to level whenever it is optimal to order. A careful examination of Theorem 2 also reveals that although we use different notation, the structure of the optimal policy is of the form $(s, S, A, p)$, as shown to be optimal for the backordering case by Chen and Simchi-Levi (2004 a). If we denote the set of inventory levels $\in (\rho_t, S_t)$ for which it is optimal to order as $A_t$, our definition of $(\rho_t, y_t^*, S_t, A_t)$ is equivalent to Chen and Simchi-Levi’s $(s_t, S_t, (s_t + S_t)/2, A_t)$, respectively (refer to §1.1). We can also generate a simple upper bound on the optimal order-up-to level $y_t^*$, utilizing our characterization of the profit function $H_t(y)$ in Lemma 4.

**Proposition 1** Let $m_t(\geq S_t)$ be the maximal $y$ such that $\Gamma_t(y, P_t(y)) \geq \Gamma_t(S_t, P_t(S_t)) - K_t$. Then, $y_t^* \in [\rho_t, m_t]$. 

11
Proof: For any $y \in [0, \rho_t)$, we have $H_t(y) < H_t(\rho_t) \leq H_t(y^*_t)$ by Lemma 4. For any $y \in (m_t, +\infty)$, note that
\[
H_t(y) = \Gamma_t(y, p_t(y)) + \alpha \int_0^{+\infty} \Pi_{t+1}(y - D_t(p_t(y))u) f_t(u) du \\
\leq \Gamma_t(y, p_t(y)) + \alpha \int_0^{+\infty} \Pi_{t+1}(y - D_t(p_t(y))u) f_t(u) du \\
< \Gamma_t(S_t, p_t(S_t)) - K_t + \alpha \int_0^{+\infty} \Pi_{t+1}(y - D_t(p_t(y))u) f_t(u) du \\
= \hat{H}_t(S_t) - K_t + \alpha \int_0^{+\infty} [\Pi_{t+1}(y - D_t(p_t(y))u) - \Pi_{t+1}(S_t - D_t(p_t(S_t))u)] f_t(u) du
\]
where $\hat{H}_t(y) = \Gamma_t(y, p_t(y)) + \alpha \int_0^{+\infty} \Pi_{t+1}(y - D_t(p_t(y))u) f_t(u) du$ as defined in the proof of Lemma 4. If $y - D_t(p_t(y))u \leq S_t$, then $\Pi_{t+1}(y - D_t(p_t(y))u) - \Pi_{t+1}(S_t - D_t(p_t(S_t))u) \leq K_{t+1}$ by (5) in the proof of Lemma 4. If $y - D_t(p_t(y))u > S_t$, we have $\Pi_{t+1}(y - D_t(p_t(y))u) - \Pi_{t+1}(S_t - D_t(p_t(S_t))u) = H_{t+1}(y - D_t(p_t(y))u) - \Pi_{t+1}(S_t - D_t(p_t(S_t))u) \leq K_t$. Thus, for any $y \in (m_t, +\infty)$,
\[
H_t(y) < \hat{H}_t(S_t) - K_t + \alpha K_{t+1} \leq H_t(S_t) \leq H_t(y^*_t).
\]
This completes the proof. 

It is worth noting that the requirement $S_t \leq S_{t+1}$ for any $t(t \leq l < T)$ is standard in the literature for non-stationary environments (e.g., Huh and Janakiraman 2005), and is not too restrictive. This will surely hold true in stationary settings, i.e., all model parameters are constant over the selling horizon. It might also be valid in other realistic business scenarios like when the price-elasticity of the product increases over time (all other parameters remaining constant) as the product progresses over its life cycle (refer to Theorem 1, item 4.).

Note that when the fixed cost is zero, $\rho_t = S_t$ and the set $A_t$ becomes empty. Consequently, we have an immediate corollary indicating that (myopic) base-stock control policy becomes optimal.

Corollary 1 Suppose $K_t = 0$ and $S_t \leq S_{t+1}$ for any $t(t \leq l < T)$. Then, a base-stock policy is optimal at the beginning of period $t$. Specifically, for any initial stock level $x \leq S_t$, it is optimal for the retailer to order up to $S_t$ and charge the price $p_t(S_t)$; otherwise, it is optimal to order nothing and charge the price $p_t(x)$. 

Likewise, we can also characterize the optimal dynamic decision policy for a stationary infinite planning horizon scenario.

Theorem 3 For the stationary infinite horizon problem, an $(s, S, p)$ policy is optimal. 

Proof: Follows from Lemma 2 and Theorem 3.5 in Huh and Janakiraman (2005).
3.3 Discussion of the Backordering Scenario

All the results of the lost sales case continue to hold even if excess demands in each period are backordered. In fact, we can then allow more general functional forms for the deterministic demand function $D_t(p)$ (demand is still multiplicative). Specifically, suppose that $D_t(p)$ is defined on $(0, P^u_t)$, and it satisfies the following assumption for any $t(1 \leq t \leq T)$:

**Assumption 2** The demand function $D_t(p)$ is positive and strictly decreasing for $p \in (0, P^u_t)$ and $\lim_{p \rightarrow P^u_t} pD_t(p) = 0$. We also assume that $p + \frac{D_t(p)}{D_t(p)}$ is strictly increasing for $p \in (0, P^u_t)$. \qed

The above assumption is rather mild and has been used previously in marketing/economics literature. A large family of demand functions used in the literature like $D_t(p) = ap^{-k_t} \gamma(a > 0, k_t > 1)$, $D_t(p) = (a - k_t p) \gamma(a, k_t, \gamma > 0)$, $D_t(p) = ae^{-k_t p}(a, k_t > 0)$, satisfies the above assumption (refer to Song et al. 2005 for more details).

Suppose we charge a backordering cost $b_t(\geq 0)$ for each unit backordered at the end of period $t$. In order to be realistic, we assume $0 \leq h_t < b_t \forall t$. The main distinguishing feature of this model compared to the lost sales scenario is in the expression of $g_t(y, p)$. For the backordering case,

$$g_t(y, p) = pD_t(p)\mu_t - h_t \Lambda_t\left(\frac{y}{D_t(p)}\right) - b_t \Theta_t\left(\frac{y}{D_t(p)}\right) - w_t y.$$ 

Without loss of generality, suppose $v^+ = v^- = v$ at the end of the planning horizon. Also, let $w_{T+1} = v$ and $\Pi_t(x) = \pi_t(x) - w_t x$ as before. The rest of the formulation and analysis then exactly follows the lost sales case with the new expression for $g_t(y, p)$. Specifically, $\Pi_t(x)$ is the same as before, $\Gamma_t(y, p)$ has an extra term $-(\alpha w_{t+1} + b_t)\Theta(\frac{y}{D_t(p)}) D_t(p)$, and $H_t(y)$ is defined in terms of the “new” $\Gamma_t(y, p)$. In the interest of space, we do not repeat the detailed derivations and proofs here.\footnote{Details are available from the authors on request.}

However, we can show that Theorem 2 is valid for the backordering scenario. That is, we can re-establish Chen and Simchi-Levi’s (2004 a) result that the optimal policy for the backordering model is also of the $(s, S, A, p)$ form, for a finite horizon problem. This immediately leads to results analogous to zero fixed cost and stationary infinite horizon cases before, i.e., Corollary 1 and Theorem 3, respectively.

We remark that although we arrive at the same conclusion as Chen and Simchi-Levi (2004 a) regarding the form of the optimal policy, the methodology we employ is quite distinct. In addition, there are some noteworthy structural differences between the results of the two papers. For example, $P^u_t$ is the “null price”, i.e., the lowest positive retail price (if any) such that $D_t(P^u_t) = 0$.\footnote{$P^u_t$ is the “null price”, i.e., the lowest positive retail price (if any) such that $D_t(P^u_t) = 0$.}
our policy parameters $\rho_t$ and $S_t$ are derived from the single-period profit function, whereas theirs ($s_t$ and $S_t$) require the evaluation of the entire profit-to-go function $H_t(y)$. Consequently, the computation of these policy parameter values might be simpler in our case.\footnote{Evidently, the optimal policy parameter values might be different in the two papers.} Furthermore, we are able to provide a more detailed characterization of the optimal profit function. In particular, the result about the behavior of $H_t(y)$ in Lemma 4, which would also be valid in the case of backordering, is not available in the previous literature. Using this result, we can generate relatively straightforward bounds on the optimal order-up-to level (analogous to Proposition 1), which again facilitates the computation of the optimal policy.

4 Conclusions and Future Research Opportunities

In this paper, we focus on a retailer facing price-sensitive stochastic demand in each period over the selling horizon. The noteworthy features of our model framework are: there is a fixed cost associated with any replenishment, excess demands might either be lost or backordered, the demand form is multiplicative, and the cost and demand parameters might be non-stationary. We determine the joint dynamic pricing and inventory control policy that maximizes the total discounted expected profit for the retailer.

Our main contribution is for the lost sales case, for which the structure of the optimal joint decision policy is not available in the literature. We show that for a finite selling horizon, the optimal policy is of the $(s, S, A, p)$ form under relatively mild conditions. Furthermore, we establish the optimality of base-stock policy when there are no fixed costs associated with orders, and $(s, S, p)$ policy when the planning horizon is infinite (with stationary parameters). We show that all these results also apply for the backordering scenario. For the backordering case, we basically reinforce the optimal policy already shown in the literature, but supplement it in terms of methodology and insights.

Two issues deserve additional attention. First, based on recent results of Huh and Janakiraman (2004), our optimal policy for the infinite horizon case will remain valid when the retailer maximizes (infinite horizon) average cost as opposed to total discounted cost. Secondly, in order to extend the optimality results of the lost sales model from an iso-elastic $D_t(p)$ to a more general $D_t(p)$ (e.g., like in Section 3.3), it is only necessary to prove Lemma 2, since all other results will hold true for more general demand forms. This is a non-trivial task which we leave for future research.
There are a number of other avenues for future research. One worthwhile direction would be to identify realistic conditions under which \((s, S, p)\) policy is optimal for the finite horizon discounted cost problem (either for lost sales or backordering case). Another interesting, but rather challenging direction, would be to incorporate the possibility of learning demand dynamically over time. Finally, extension to a decentralized and/or competitive setting would be a worthwhile research endeavor.

**Appendix: Proofs of Lemmas and Theorems**

For exposition purposes, in all the proofs in the appendix, we suppress the subscript \(t\).

**Proof of Theorem 1:**

Partial differentiation of \(\Gamma(y, p)\) with respect to \(p\) and \(y\) yields:

\[
\frac{\partial \Gamma(y, p)}{\partial p} = D(p) \int_0^{\frac{y}{D(p)}} uf(u)du \left\{ V\left(\frac{y}{D(p)}\right) - k\frac{p-v}{p}\right\},
\]

\[
\frac{\partial \Gamma(y, p)}{\partial y} = p[1 - F\left(\frac{y}{D(p)}\right)] + vF\left(\frac{y}{D(p)}\right) - w.
\]

By Theorem 1 in Song et al. (2005), for any given order-up-to inventory level \(y\) there exists a unique retail price \(P(y)\) such that \(\Gamma(y, p)\) is maximized, and \(\Gamma(y, P(y))\) is concave. From the above equations, it is clear that this \(P(y)\) satisfies

\[
\frac{kv}{P(y)} - k + V\left(\frac{y}{D(P(y))}\right) = 0. \tag{6}
\]

Based on (6), we have \(P(y) > v\) for any \(y \in (0, +\infty)\). Furthermore, we also know the following:

1. The maximizer of \(\Gamma(y, P(y))\) is unique and we denote it as \(S\).

2. \(P(y)\) is strictly decreasing on \((0, +\infty)\).

3. Let \(Z(y) = \frac{y}{D(P(y))}\) be the stocking factor for any given \(y\). Then \(Z(y)\) is strictly increasing on \((0, +\infty)\).

Let \(s(\leq S)\) be the maximal inventory level such that \(\Gamma(s, P(s)) \leq \Gamma(S, P(S)) - K\). Then, \(\Gamma(x) = \Gamma(S, P(S)) - K + wx\) for any \(x \leq s\) and \(\Gamma(x) = \Gamma(x, P(x)) + wx\) for any \(x > s\). If \(x \leq s\), as \(\Gamma(x)\) is linear and increasing, we get \(\Gamma'(x) + kx\Gamma''(x) \geq 0\). If \(x > s\), then

\[
\Gamma'(x) + kx\Gamma''(x) = [P(x) + kxP'(x)]U'(Z(x)) + (v - h)F(Z(x)) + vkxf(Z(x))Z'(x).
\]
As $P(x)$ satisfies $\frac{kv}{P(x)} \int_0^{Z(x)} uf(u)du + U(Z(x)) = 0$, taking derivative with respect to $x$ and substituting it into the above equation, we have

$$\Gamma'(x) + kx\Gamma''(x) = v[F(Z(x)) + kD(P(x))\frac{P'(x)}{P(x)} \int_0^{Z(x)} uf(u)du].$$

As $P'(x) = \frac{-V'(Z(x))\frac{1}{P'(x)}}{-\frac{kv}{P(x)} + V'(Z(x))\frac{kZ(x)}{P'(x)}}$, we obtain $D(P(x))\frac{P'(x)}{P(x)} = -\frac{V'(Z(x))}{-\frac{kv}{P(x)} + kV'(Z(x))Z(x)}$. Substituting this into the right hand side of the above equation, in order to prove $\pi'(x) + kx\pi''(x) \geq 0$, it is sufficient to show that

$$-\frac{kv}{P(x)} F(Z(x)) + kV'(Z(x))\{Z(x)F(Z(x)) - \int_0^{Z(x)} uf(u)du\} \leq 0.$$ 

This is obviously true as $zF(z) - \int_0^z uf(u)du = \Lambda(z) \geq 0$ (note that $V(z)$ is strictly decreasing as per Lemma 1).

Finally, we show the result of part (4). Let $z = \frac{y}{p}$ and we rewrite the expected profit in terms of $(z, p)$ rather than $(y, p)$. Hence, the expected profit of the single period model is

$$\Omega(z, p) = pD(p)[\mu - \Theta(z)] + vD(p)\Lambda(z) - wD(p)z.$$ 

Taking partial derivative of $\Omega(z, p)$ with respect to $p$, we get $\frac{\partial \Omega(z, p)}{\partial p} = \frac{D(p)}{p}\{(1 - k)p[\mu - \Theta(z)] - kv\Lambda(z) + kwz\}$. Thus, for any given $z$ there exists a unique $p(z)$ such that $\Omega(z, p)$ is maximized and it can be expressed as

$$p(z, k) = \frac{k}{k - 1}[(w - v)\frac{z}{\mu - \Theta(z)} + v].$$

By $p(z, k)$, the expected single period profit can be rewritten as a one variable function $\Omega(z, p(z, k))$ and its first derivative is

$$\frac{d\Omega(z, p(z, k))}{dz} = D(p(z, k))[p(z, k) - w - [p(z, k) - v]F(z)].$$

The optimal stocking factor $z(k)$ for the given $k$ should satisfy $p(z, k) - w - [p(z, k) - v]F(z) = 0$, i.e.,

$$(w - v)[1 - \frac{k}{V(z)}] + v[1 - F(z)] = 0,$$

based on the expression of $p(z, k)$. It is then obvious that the solution of $z(k)$ is unique. Taking derivative with respect to $k$ on both sides in the above equation, we get

$$z'(k) = \frac{(w - v)V(z(k))}{(w - v)kv^2(z(k)) - vV(z(k))^2} < 0.$$ 

Note that $p(z, k)$ is increasing in $z$. This implies that the optimal retail price $p(k) = p(z(k), k)$ is decreasing in price elasticity $k$. 

16
Now let us study the behavior of the optimal base-stock level $S(k) = z(k)p(k)^{-k}$ in terms of the product elasticity $k$. Suppose that the current price elasticity value is $k_0(> 1)$. We will show that $S(k)$ is increasing on $[k_0, +\infty)$. Without loss of generality, we assume that $p(k) < 1$ for any $k \in [k_0, +\infty)$ based on the expression of $p(k)$ (by scaling down the units of $w$ and $v$). As $S'(k) = \{z'(k)p(k)^k - z(k)p(k)^k[lnp(k) + k(p'(k))]p(k)^{2k}\}p(k)^{2k}$ and $ln[p(k)] < 0$ by the above assumption, in order to show $S'(k) \geq 0$ it is sufficient to show

$$z'(k) \geq kz(k)\frac{p'(k)}{p(k)}.$$ \hspace{1cm} \text{(7)}

From the equation satisfied by $z(k)$, we get $v = \frac{k - V(z(k))}{[1 - F(z(k))V(z(k))]}(w - v)$. Substitution of this expression of $v$ into $p(k)$ and $z'(k)$, we get $p(k) = \frac{k}{V(z(k))[1 - F(z(k))]}(w - v)$ and

$$z'(k) = \frac{[1 - F(z(k))V(z(k))]}{kV'(z(k))[1 - F(z(k))] - kV(z(k)f(z(k)) + f(z(k))V(z(k))^2}.$$n

If we substitute the above two expressions in (7), it simplifies to $r(z(k))V(z(k)) \geq 1$, which proves the result. \hfill \Box

**Proof of Lemma 2:**

For simplicity, let $v = \alpha w_{t+1} - h_t$ and we suppress the subscript $t$ from all notations. For any given order-up-to inventory level $y(\geq 0)$, recall from Theorem 1 that $P(y)$ is the corresponding optimal retail price. Let $P'(y)$ be the unique positive solution of $V(\frac{y}{p'(y)}) = k - l$ for a given constant $l$, if any. Furthermore, let $C = \{(y, P(y))|y \geq 0\}$ and $C' = \{(y, P'(y))|y \geq 0\}$. The common point on both $C$ and $C'$, if any, is denoted by $(y', p')$. We can see that both $P(y)$ and $P'(y)$ are decreasing, but $P'(y)$ is decreasing faster than $P(y)$ at the common point $(y', p')$. Note from Theorem 1 that

$S$ is the maximizer of $\Gamma(y, P(y))$ on $[0, +\infty)$.

For any given $S \leq y^1 < y^2$ and retail price $p^2$, based on the relationship between $p^2$ and $P(y^2)$, we consider the following two cases:

**Case 1** $p^2 \geq P(y^2)$:

In this case, we choose $p^1 = P(y^1)$. By Theorem 1, the requirements in Lemma 2 are satisfied.

**Case 2** $p^2 < P(y^2)$:

In this case, let $l_2$ be the constant such that $(y^2, p^2)$ is on $C^{l_2}$. Based on the sign of $l_2$ and the relationship between $y^1$ and $y^2$, we consider the following three subcases:

**Subcase 2(a)** $l_2 > 0$ and $y^1 \leq y^2$:
In this subcase, let \( p^1 = P(y_1) \). It is obvious that \( \Gamma(y^1, p^1) \geq \Gamma(y^2, p^2) \geq \Gamma(y^2, P(y^2)) \geq \Gamma(p^2, y^2) \). We also know that the leftover of \((y^2, p^2)\) is larger than the one of \((y^1, p^1)\). Thus, it is sufficient to show that the leftover of \((y^2, p^2)\) is larger than the one of \((y^2, p^2)\). Note that for any point \((y, p)\) on \(C^{l_2}\), we define \( z = \frac{y}{D(p)} \) and it is a constant. Hence, the leftover, \( D(p)[z - u] \), of \((y, p)\) is decreasing in terms of \( p \). This proves Subcase 2(a).

**Subcase 2(b) \( l_2 > 0 \) and \( y^1 > y^2\):**

Recall that \( l_2 \) is the constant such that \((y^2, p^2)\) is on \(C^{l_2}\). Let \( l_1 \) be the constant such that \((y^1, P(y^1))\) is on \(C^{l_1}\), and suppose \((y^1, p^1)\) be the point on both \(C^{l_2}\) and \( \{(y, p)|y = y^1\} \).

For a fixed \( y^i (i = 1 \text{ or } 2) \), the retail price \( p^i(l) \) of the point \((y^i, p)\) on \(C^d (l_2 \leq l \leq l_1)\) is a function of \( l \) and

\[
V(\frac{y^i}{D(p^i(l))}) = k - l.
\]

Taking derivative with respect to \( l \) on both sides, we get

\[
\frac{dp^i(l)}{dl} = -\frac{p^i(l)D(p^i(l))}{k y^i} [V'(\frac{y^i}{D(p^i(l))})]^{-1}.
\]

We need to study the behavior of \( \Gamma(y^i, p^i(l)) \) with respect to the curve parameter \( l \). As \( \frac{d\Gamma(y^i, p^i(l))}{dl} = \frac{\partial\Gamma(y, p)}{\partial p} \{y = p^i(l)\} \frac{dp^i(l)}{dl} = D(p^i(l)) \int_0^{\frac{y^i}{D(p^i(l))}} u f(u) du [kv - lp^i(l)] - \int_0^{\frac{1}{D(p^i(l))}} D(p^i(l)) \frac{1}{V'(\frac{y^i}{D(p^i(l))})^0} \frac{dV'(\frac{y^i}{D(p^i(l))})}{dl}] \geq 0 \) and \( \frac{d\Gamma(y^i, p^i(l))}{dl} \) is a constant for any fixed \( l \), we get \( \frac{d\Gamma(y^i, p^i(l))}{dl} \) is decreasing in terms of \( p^i(l) \). Thus, let \((y^2, \tilde{p})\) be the point on \(C^{l_1}\). Then we have

\[
\Gamma(y^1, P(y^1)) - \Gamma(y^1, p^1) = \int_{l_2}^{l_1} d\Gamma(y^1, p^i(l)) \leq \int_{l_2}^{l_1} d\Gamma(y^2, p^i(l)) = \Gamma(y^2, \tilde{p}) - \Gamma(y^2, p^2).
\]

On the other hand, by the argument in Subcase 2(a), we know

\[
\Gamma(y^1, P(y^1)) \geq \Gamma(y^2, \tilde{p}).
\]

Combining the above two inequalities, we get \( \Gamma(y^1, p^1) \geq \Gamma(y^2, p^2) \). Following the same argument as Subcase 2(a), it is obvious that the leftover of the point \((y^2, p^2)\) is larger than or equal to the leftover of the point \((y^1, p^1)\). This proves the Subcase 2(b).

**Subcase 2(c) \( l_2 \leq 0 \):**

In this subcase, there is no any common point on both \(C\) and \(C_{l_2}\) and we first study the behavior of \( \Gamma(y, p) \) on \(C^d\) for any \( l \leq 0 \).

Let \( Z(l) \) be the constant such that \( V(Z(l)) = k - l \). Then, \( \Gamma(y, p) \) on \(C^d\) can be expressed as

\[
\Gamma(p) = pD(Z(l))[\mu - \Theta(Z(l))] + v\Lambda(Z(l))D(Z(l)) - wZ(l)D(Z(l)).
\]
It is obvious that $\Gamma(p)$ is unimodal and the maximizer can be expressed as

$$p(Z(l)) = \frac{k}{k-1} \frac{wZ(l) - v\Lambda(Z(l))}{\mu - \Theta(Z(l))}.$$ 

Its first derivative with respect to $Z(l)$ is

$$\frac{dp(Z(l))}{dZ(l)} = \frac{k}{k-1} \frac{(w-v)\int_0^{Z(l)} uf(u)du}{|\mu - \Theta(Z(l))|^2}.$$

Let $y(Z(l)) = Z(l)D(p(Z(l)))$. Its first derivative with respect to $Z(l)$ can be expressed as

$$\frac{dy(Z(l))}{dZ(l)} = \frac{D(Z(l))}{V(Z(l))} \left\{ 1 - kZ(l) \frac{p'}{p} \right\} = \frac{D(Z(l))}{V(Z(l))} \left\{ V(Z(l)) - k \frac{Z(l)(w-v)}{(w-v)Z(l) + v[\mu - \Theta(Z(l))]} \right\}$$

$$\geq \frac{D(Z(l))}{V(Z(l))} \left\{ V(Z(l)) - k \right\} = \frac{D(Z(l))}{V(Z(l))} (l) \geq 0.$$

As $Z(l)$ is increasing with respect to $l$, $y(Z(l))$ is increasing with respect to $l$ on $(-\infty, 0]$. Note that from the argument in Subcase 2(b), we get $y(Z(0)) \leq S$. Thus, for any $l \in (-\infty, 0]$, we have $y(Z(l)) \leq S$. Combining this and the unimodal property of $\Gamma(p)$, we get $\Gamma(y_1, p_1) \geq \Gamma(y_2, p_2)$. \hfill $\Box$

**References**


