Optimal Dynamic Joint Pricing and Inventory Control for Multiplicative Demand with Fixed Order Costs and Lost Sales

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Abstract

In this paper we study the optimal dynamic decision-making problem for a retailer selling a single product and facing price-sensitive, stochastic (multiplicative) demand in each period. Any leftover inventory for a period is carried over and charged a holding cost at the end of the period. Leftovers at the end of the selling horizon earn a non-negative salvage value. On the other hand, any unsatisfied demand, and associated revenue, is lost to the system. The purchasing cost for the retailer includes both variable and fixed components. The objective of the retailer is to maximize its discounted expected profit over the selling horizon by dynamically deciding on the optimal pricing and inventory control policy for each period. Under some mild conditions on the model parameters, we characterize the structure of the retailer’s optimal inventory policy, and specify the optimal price to charge in each period, for both finite and infinite selling horizons.

Key Words: Dynamic pricing and inventory control, Multiplicative price-sensitive stochastic demand, Fixed order cost, Lost sales.
1 Introduction and Literature Overview

In this paper, we study the optimal dynamic decision-making problem for a retailer selling a single product. We address in particular the two most strategic decisions a retailer is faced with - those of setting price and planning for the amount of inventory to procure (Stern and El-Ansary 1992). The issue of dynamic inventory control has been extensively studied in the operations management (OM) literature (Portues 2002). These models normally assume the retail price per unit to be constant over the planning horizon. However, recent advances in information technology have made it cheaper to change prices dynamically, especially in online environments. A number of retailers (e.g., Dell, Amazon.com) are already using dynamic pricing as a competitive strategy, and some of them (e.g., Dell) are even altering both pricing and inventory decisions regularly in order to increase profits (Chen and Simchi-Levi 2004b). This has prompted an academic interest in determining the optimal dynamic joint policy in terms of these two decisions (Elmaghraby and Keskinocak 2003).

The common framework adopted for such research is a periodic-review setting with the end customer demand in each period being both stochastic and price-sensitive. We also adopt such a setting, but assume specifically that the demand in each period can be expressed as the product of a price-sensitive deterministic function and a non-price-sensitive positive random variable (multiplicative demand). This demand form has been widely used in the literature and there is empirical evidence supporting its realism (Cachon and Kok 2006, Petruzzi and Dada 1999, and references therein). At each review epoch, the retailer needs to decide on the replenishment quantity from the manufacturer (which is received instantaneously), and the price to charge to the end customers for that period. Demands which cannot be satisfied promptly due to insufficient inventory are lost, and result in lost revenue.\(^1\) The lost sales assumption is particularly suitable for situations in which customers are unwilling to wait for their demands to be filled (as in most retail settings). Any leftover inventory from a period is carried forward to satisfy future demand, and a holding cost is charged proportional to the inventory level at the end of the period. The leftovers at the end of the selling horizon are salvaged at a nonnegative per unit value. The retailer’s replenishment cost, on the other hand, involves two elements - a variable cost per unit and a fixed order cost for any positive purchase quantity. Note that the analysis in this paper is based on non-stationary demand and cost parameters, albeit subject to certain restrictions

\(^1\)We briefly discuss the alternate scenario in which excess demands are backordered in §3.
(e.g., decreasing fixed order costs over time, no speculative motive for holding inventory). Our objective is to determine the optimal dynamic joint pricing and inventory control policy that would maximize the retailer’s total discounted expected profit over the selling horizon, which can be either finite or infinite.

Such dynamic analysis has recently become popular in the OM literature. Most of this literature can be categorized based on two primary characteristics: whether the excess demand in each period is backordered or lost, and whether the price-sensitive stochastic demand is of the multiplicative or of the additive form (i.e., price-sensitive deterministic function plus a non-price-sensitive positive random variable). A classification of the relevant research based on these characteristics is provided in Tables 1 and 2. Table 3 provides a summary and highlights the open research area.

<table>
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<th>Demand Form</th>
<th>Riskless/Random</th>
<th>Fixed Cost</th>
<th>Optimal Policy</th>
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Table 1: **Dynamic Joint Inventory Control and Pricing Models With Backordering**

<table>
<thead>
<tr>
<th>Demand Form</th>
<th>Riskless/Random</th>
<th>Fixed Cost</th>
<th>Optimal Policy</th>
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<tbody>
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<tr>
<td>additive</td>
<td>general/general</td>
<td>positive</td>
<td>(s, S, p)</td>
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Table 2: **Dynamic Joint Inventory Control and Pricing Models With Lost Sales**

For additive demand, it is well-known that an (s, S, p) policy is optimal. That is, at the beginning of each period an (s, S) policy is optimal for replenishment, and the value of the optimal
price $p$ depends on the post-replenishment inventory level. This result has been established for backordering and lost sales models by Chen and Simchi-Levi (2004 a, b) and Chen et al. (2006), respectively. Using an alternative approach, Huh and Janakiraman (2006) also prove the optimality of the $(s, S, p)$ policy for both cases under quite general conditions (details below).

The optimal policy for multiplicative demand is more involved. If excess demands are back-ordered, Chen and Simchi-Levi (2004 a, b) have shown that an $(s, S, p)$ policy is optimal for the infinite horizon problem, but is not necessarily so when the problem horizon is finite. In the finite horizon scenario, the total profit function, at least for a certain range of stock levels, might not be well-behaved. Using an elegant concept called symmetric $K$-convexity, they have shown that the optimal policy is of the $(s, S, A, p)$ form in that case. Under this policy, if the starting inventory of a period is:

(i) less than $s$ or belongs to set $A (A \subset [s, (s + S)/2])$, then the optimal order-up-to level is $S$. Note that $A$ is the set of all inventory levels within $[s, (s + S)/2]$ for which ordering is optimal. This set is not known a-priori. It needs to be determined via some search, and can well be empty.

(ii) greater than $(s + S)/2$ or belongs to set $[s, (s + S)/2] \setminus A$, then no order is placed.

The optimal $p$ depends on the post-replenishment inventory level. To the best of our knowledge, the structure of the optimal policy when excess demands are lost remains an open question. According to Polatoglu and Sahin (2000), multiple optimal order-up-to levels might exist (also refer to Chen and Simchi-Levi, 2004a, pg 892).

Motivated by the above, we focus on determining the optimal policy for a modelling paradigm with multiplicative demand, lost sales and a fixed cost associated with any replenishment (i.e., the open question). Our primary contribution is to establish the optimality of the $(s, S, A, p)$ policy for a finite selling horizon in this paradigm. We prove this result by using an approach different from the one adopted by Chen and Simchi-Levi (2004a) for the backordering scenario. However, as discussed in §3, our technique readily applies to the backordering model, re-establishing Chen and Simchi-Levi’s (2004a) optimal policy result. Moreover, we show that for the special case with zero fixed order costs, the optimal policy reduces to a base-stock policy, whereas for the stationary infinite horizon case, an $(s, S, p)$ policy is optimal.\footnote{Note that we characterize the optimal inventory policy in this paper; as far the optimal pricing policy is concerned, we only specify its value in each period based on the post-replenishment inventory level, as is the}
Our optimality proofs borrow some concepts and results from Huh and Janakiraman (2006; henceforth H&J). Hence, we briefly discuss that paper before presenting our results. H&J first identify a condition (“Condition 1”) related to the single period expected profit function. They show that if this condition holds, then \((s, S, p)\) policy is optimal for stationary, infinite horizon problems. Subsequently, they establish a more restrictive “Condition 2” (related also to the single period expected profit function) which is sufficient for the optimality of \((s, S, p)\) policy for finite horizon problems. However, the conditions need to be studied for specific scenarios in order to verify their validity. This verification process is non-trivial. H&J are able to show that “Condition 1” is valid for: i) backordering models for both additive and multiplicative demand forms, and ii) additive, lost sales models. They also verify that “Condition 2” holds true for additive demand models (lost sales and backordering). However, they do not provide any results for the lost sales model under multiplicative demand, which is the focus of this paper.

We first show a key property which implies that H&J’s “Condition 1” is valid for this case. This enables us to prove the form of the optimal inventory policy (no-ordering property) for high initial stock levels in case of finite horizon problems, and optimality of the \((s, S, p)\) policy for (stationary) infinite horizon scenarios. More importantly, we establish a new property of the expected profit function from which we derive the optimal inventory policy for low initial stock situations (unique order-up-to level property) in finite horizon cases. Putting these together, we characterize the optimal inventory policy for our modelling paradigm, and also specify the optimal price to charge in each period. Our new property is different compared to the results in H&J, except that it also requires the single period expected profit function to exhibit certain properties. However, these requirements are less restrictive compared to “Condition 2” of H&J. Specifically, while H&J require certain properties on a plane, we only put restrictions on the behavior on a curve.\(^3\) Furthermore, although we use dynamic programming (DP) approach in our paper, we do not utilize concepts like \(K - convexity\), which is typical in the literature for such methodology (e.g., Chen and Simchi-Levi 2004a).

The remainder of this paper is organized as follows. In §2, we present our model framework. §3 develops the optimal policy results. We end the paper with our concluding remarks in §4.

\(^3\)However, “Condition 2” of H&J is sufficient for optimality of the \((s, S, p)\) policy, while we establish optimality of the \((s, S, A, p)\) policy.
2 Model Framework

Suppose that the periodic-review (finite) selling horizon has $T$ periods indexed forward by period index $t$, $1 \leq t \leq T$. If the retail price charged in period $t$ is $p$, we assume the multiplicative price-sensitive demand in that period to be $D_t(p)\epsilon_t$, where $D_t(p)$ is a strictly decreasing,\footnote{Throughout the paper we use “increasing” and “decreasing” in the weak sense, unless otherwise stated.} deterministic function defined on $(0, P_u^t)$\footnote{We define $P_u^t$ as the lowest positive retail price such that $D_t(P_u^t) = 0$, i.e., “null price”. If $D_t(p) > 0$ for all $p > 0$, then we let $P_u^t = +\infty$.} and $\epsilon_t$ is a random variable defined and positive on $(L_t, U_t)$, $0 \leq L_t < U_t \leq \infty$. Without loss of generality, we assume that $E[\epsilon_t] = 1$. Let $f_t(u)$ and $F_t(u)$ be the density and distribution functions of $\epsilon_t$, respectively. Note that demands for different periods need to be independent in our setting, but not necessarily be identically distributed. We also assume that $\lim_{u \downarrow L} f_t(u) > 0$ if $L_t = 0$ and $f_t(u) = 0$ for any $u \in [0, L_t] \cup [U_t, +\infty)$. The retailer’s problem can be described as follows. Given an initial stock level $x$ ($\geq 0$) before ordering at the beginning of period $t$, the retailer needs to decide on the order-up-to inventory level $y$ ($\geq x$), or equivalently the order quantity $(y - x)$, and the retail price $p$, before any demand is realized. The objective is to maximize the total discounted expected profit from period $t$ until the end of the planning horizon $T$. For an order of $(y - x)$ from the manufacturer, the retailer’s replenishment cost is given by $K_t\delta(y - x) + w_t(y - x)$, where $w_t$ ($\geq 0$) is the per unit purchasing cost, $K_t$ ($\geq 0$) is the fixed order cost, and $\delta(y - x) = 1$ if $y > x$ and $0$ otherwise. Once the order is placed, it is received immediately by the retailer, after which the demand in period $t$ is realized. The demand depends on $p$ and the realization of $\epsilon_t$.

Let $\pi_t(x)$ be the optimal discounted expected total profit from period $t$ until the end of the planning horizon $T$, when the starting inventory level in period $t$ is $x$. We define $g_t(y, p)$ as the expected profit in period $t$ when $x = 0$ and $K_t = 0$. We also define $LO_t(y, p, u)$ as the leftover inventory level at the end of period $t$ given the order-up-to inventory level $y$, the retail price $p$, and the realization $u$ of $\epsilon_t$ in period $t$. If $0 < \alpha \leq 1$ denotes the (given) discount factor, then the following DP formulation represents the retailer’s maximization problem:

$$
\pi_t(x) = \max_{\{p>0, y\geq x\}} \{-K_t\delta(y - x) + g_t(y, p) + \alpha \int_0^{+\infty} \pi_{t+1}(LO_t(y, p, u))f_t(u)du\} + w_tx. \quad (1)
$$

Note that if $v$ ($\geq 0$) represents the terminal salvage value, then $\pi_{T+1}(x) = vx$ for any $x \geq 0$ and $\pi_{T+1}(x) = 0$ otherwise.
Tractable analysis of the above problem requires the deterministic part ($D_t(p)$) and the random part ($\epsilon_t$) of the demand to have the following properties.

**Assumption 1** $D_t(p)$ is positive and strictly decreasing on $(0, P_t^u)$ and $\lim_{p \to P_t^u} p D_t(p) = 0$. We also assume $D_t(p)$ is continuously differentiable and the elasticity $\eta(p) = -\frac{D_t(p)}{D_t'(p)} (> 0)$ of the demand function is increasing for $p \in (0, P_t^u)$. Furthermore, $\frac{D_t(p)}{D_t'(p)}$ is monotone and concave, while $p + \frac{D_t(p)}{D_t'(p)}$ is strictly increasing for $p \in (0, P_t^u)$.

Assumption 1 basically implies that the curvature of $D_t(p)$, defined as $E(p) = D_t(p) \frac{D_t''(p)}{D_t'(p)^2}$, should not be highly positive and it should increase in $p$. For example, the concavity of $\frac{D_t(p)}{D_t'(p)}$ and the strictly increasing property of $p + \frac{D_t(p)}{D_t'(p)}$ are equivalent to $E'(p) \geq 0$ and $E(p) < 2$, respectively. Assumption 1 is quite established in OM/economics literature (refer to Cowan 2004, Ziya et al. 2004, Song et al. 2006, for details). In fact, most of the common demand functions satisfy the assumption. Examples include concave functions ($D_t(p) = a_t - p^{k_t}$ ($a_t > 0, k_t > 1$), $D_t(p) = (a_t - k_t p)^{\gamma_t}$ ($a_t > 0, k_t > 0, 0 < \gamma_t < 1$), $D_t(p) = a_t - e^{k_t p}$ ($a_t, k_t > 0$)), as well as convex ones ($D_t(p) = a_t p^{-k_t}$ ($a_t > 0, k_t > 1$), $D_t(p) = (a_t - k_t p)^{\gamma_t}$ ($a_t > 0, k_t > 0, \gamma_t \geq 1$ or $a_t > 0, k_t < 0, \gamma_t < -1$), $D_t(p) = a_t e^{-k_t p}$ ($a_t > 0, k_t > 0$)).

**Assumption 2** $r_t(u) = \frac{u f_t(u)}{1 - F_t(u)}$ is increasing in $u$ on $(L_t, U_t)$ for any $t$ ($1 \leq t \leq T$).

Assumption 2 implies an increasing generalized failure rate (IGFR) for $\epsilon_t$, and is a mild requirement satisfied by distributions like Uniform, Gamma with shape parameter $\geq 1$, Beta with both parameters $\geq 1$, Weibull with shape parameter $\geq 1$, Normal, Exponential, and Left-truncated (at 0) Logistic (refer to Lariviere 2006 for more details).

Let $\Lambda_t(z) = \int_0^z (z - u) f_t(u) du$ and $\Theta_t(z) = \int_z^{+\infty} (u - z) f_t(u) du$ for any $z \in (0, \infty)$ represent the relative overage and underage functions, respectively (as in Petruzzi and Dada 1999). Furthermore, let $V_t(z) = \frac{1 - \Theta_t(z)}{\int_0^{x(z)} u f_t(u) du}$ for any $z \in (L_t, U_t)$. We can establish the following basic properties of $V_t(z)$ (refer also to Song et al. 2006):

**Lemma 1** $V_t(z)$ is strictly decreasing on $(L_t, U_t)$, $\lim_{z \to L_t} V_t(z) = +\infty$, and $\lim_{z \to U_t} V_t(z) = 1$. Furthermore, $\Lambda_t(z)$ and $\Theta_t(z)$ can be rewritten as:

$$\Lambda_t(z) = z F_t(z) - \int_0^z u f_t(u) du \quad \text{and} \quad \Theta_t(z) = 1 - z [1 - F_t(z)] - \int_0^z u f_t(u) du. \quad (2)$$
Proof: Note that $V_t(z) = (1 - \frac{z(1 - F_t(z))}{\Theta_t(z)})^{-1} = \frac{1 - \Theta_t(z)}{\int_0^z u f_t(u) du}$. In order to prove the lemma, we then need to show that $V_t'(z) = (1 - F_t(z)) \int_0^z u f_t(u) du - (1 - \Theta_t(z)) z f_t(z) < 0$. As $1 - \Theta_t(z) = \int_0^z u f_t(u) du + z[1 - F_t(z)]$ (refer to (2)), the above inequality is equivalent to $\rho(z) = [1 - g_t(z)] \int_0^z u f_t(u) du - z^2 f_t(z) < 0$. Since $\rho(z) \mid_{z=L_t} = -L_t^2 f_t(L_t) \leq 0$, it is sufficient to study the behavior of $\rho(z)$ for $z \in (L_t, U_t)$ for proving the lemma. Differentiating $\rho(z)$ we have,

\[ \rho'(z) = -z f_t(z) - z f_t(z) g_t(z) - z^2 f_t'(z) - g_t'(z) \int_0^z u f_t(u) du \]

\[ = -z[1 - F_t(z)] g_t'(z) - g_t'(z) \int_0^z u f_t(u) du = -[1 - \Theta_t(z)] g_t'(z). \]

From Assumption 2 we know that $g_t'(z) \geq 0$ and $g_t'(0^+) > 0$ if $L_t = 0$ (recall that $\lim_{u \to 0} f_t(u) > 0$ if $L_t = 0$). We consider two cases based on the value of $L_t$. If $L_t > 0$, then $\rho(L_t) < 0$ and $\rho'(z) \leq 0$ for any $z \in (L_t, U_t)$; hence, $\rho(z) < 0$ for any $z \in (L_t, U_t)$. If $L_t = 0$, then $\rho(0) = 0$ and $g_t'(0^+) > 0$, which implies that $\rho'(z) < 0$ for any $z \in (0, U_t)$, i.e., $\rho(z) < 0$ for any $z \in (L_t, U_t)$. Therefore, $V_t'(z) < 0$ for any $z \in (L_t, U_t)$. The limits of $V_t(z)$, on the other hand, follow from our assumption about $f_t(u)$, while the two identities in (2) are obtained by simplifying and rearranging terms in the original definitions of $\Theta_t(z)$ and $\Lambda_t(z)$.

3 Model Analysis

We assume that the only penalty incurred by the retailer because of lost sales is the foregone revenue, and there is a linear holding cost charged (at the rate $h_t$/unit) when the leftover inventory $LO_t(y, p, u) = \max \{0, y - D_t(p) u\}$ is carried over to the next period. The exact expression of $g_t(y, p)$ will then be:

\[ g_t(y, p) = p D_t(p) [1 - \Theta_t(\frac{y}{D_t(p)})] - h_t \Lambda_t(\frac{y}{D_t(p)}) D_t(p) - w_t y. \]

In the above expression, the first term denotes the total expected revenue, the second term represents the expected holding cost, and the last term is the (variable) replenishment cost. Note that since any excess demand is lost, we define $\pi_t(x) = \pi_t(0)$ if $x < 0$.

Define $w_{T+1} = v$. Also, for any $t$ $(1 \leq t \leq T + 1)$, let $\Pi_t(x) = \pi_t(x) - w_t x$ for $x \geq 0$ and $\Pi_t(x) = \Pi_t(0)$ otherwise. The DP formulation in (1) can then be equivalently stated as:

\[ \Pi_t(x) = \max_{p, y \geq x} \{ -K_t \delta(y - x) + \Gamma_t(y, p) + \alpha \int_0^\infty \Pi_{t+1}(LO_t(y, p, u)) f_t(u) du \}, \quad (3) \]
for any \( t \) (\( 1 \leq t \leq T \)), where \( \Pi_{T+1}(x) = 0 \) for any \( x \) and

\[
\Gamma_t(y, p) = pD_t(p)[1 - \Theta_t\left(\frac{y}{D_t(p)}\right)] + (\omega t + 1 - h_t)\Lambda_t\left(\frac{y}{D_t(p)}\right)D_t(p) - w_t y.\tag{4}
\]

\( \Gamma_t(y, p) \) is the expected total profit function for a price-setting newsvendor with a salvage value of \((\omega t + 1 - h_t)\) (say, \( v_t \)) and zero initial stock. We assume \( v_t \geq 0 \) and \( v_t < w_t \) throughout this paper. This basically implies that it is cheaper to procure a unit than to carry it over from the previous period, eliminating the “speculative” motive for holding inventory. Note also that

\[
\frac{\partial \Gamma_t(y, p)}{\partial p} = D_t(p) \int_0^{\frac{y}{D_t(p)}} u f_t(u) du \left\{ V_t\left(\frac{y}{D_t(p)}\right) + (p - v_t)\frac{D_t'(p)}{D_t(p)}\right\},
\]

\[
\frac{\partial \Gamma_t(y, p)}{\partial y} = p\left[1 - F_t\left(\frac{y}{D_t(p)}\right)\right] + v_t F_t\left(\frac{y}{D_t(p)}\right) - w_t.
\]

In the sequel, we utilize the above expressions to analyze the single period profit function, and based on those results, characterize the optimal policy for the multi-period case.

### 3.1 Analysis of the Single Period Model

Let us denote the particular period as \( t \). For an initial inventory level \( x \), the retailer’s profit for period \( t \) is

\[-K_t \delta(y - x) + \Gamma_t(y, p) + w_t x,\]

where \( \Gamma_t(y, p) \) is given by (4). If \( \frac{y}{D_t(p)} \leq L_t \) for some pair \((y, p)\), then \( \Gamma_t(y, p) \) can be simplified as \((p - w_t) y\) and it is always increasing in terms of \( p \). That is, for any pair \((y, p)\) satisfying \( \frac{y}{D_t(p)} \leq L_t \), there exists a pair \((y, \hat{p})\) such that \( \frac{y}{D_t(p)} > L_t \) and \( \Gamma_t(y, \hat{p}) > \Gamma_t(y, p) \). Hence, in the remainder of the paper we always assume \( \frac{y}{D_t(p)} > L_t \). For any given order-up-to inventory level \( y \in [x, +\infty) \) at the beginning of period \( t \), the corresponding feasible range of \( p \) is \( (D_t^{-1}\left(\frac{y}{L_t}\right), P_t^u) \).\(^6\)

The next proposition summarizes the properties of \( \Gamma_t(y, p) \) and the optimal policy characteristics for the single period model.

**Proposition 1** Given any order-up-to inventory level \( y \geq x \) for period \( t \), there exists a unique \( P_t(y) \), solution of \( \frac{\partial \Gamma_t(y, p)}{\partial p} = 0 \), such that \( \Gamma_t(y, p) \) is maximized. Moreover, \( P_t(y) \) is strictly decreasing and \( \Gamma_t(y, P_t(y)) \) is strictly concave in \( y \). Furthermore, the following results hold true:

\(^6\)If \( L_t = 0 \), we define \( D_t^{-1}\left(\frac{y}{L_t}\right) = 0 \).
1. Let $s_t$ be the unique maximizer of $\Gamma_t(y, P_t(y))$ and $s_t (\leq S_t)$ be the maximal inventory level such that $\Gamma_t(s_t, P_t(s_t)) \leq \Gamma_t(S_t, P_t(S_t)) - K_t$. If there is no such $s_t$, define $s_t = 0$. Then, an $(s_t, S_t, P_t)$ policy is optimal for the retailer in period $t$.

2. Let $Z_t(y) = \frac{y}{D_t(P_t(y))}$ for any $y > 0$. Then, $Z_t(y)$ is increasing in $y$, and hence, so is the leftover at the end of the period i.e., $LO_t(y, p, u) = \max\{y - D_t(P_t(y))u, 0\}$, for any realization $u$ of $\epsilon_t$.

**Proof:** For notational simplicity, we suppress the subscript $t$ in the following proof. For any given order-up-to inventory level $y \in [x, +\infty)$ at the beginning of period $t$, let $P(y) \in (D^{-1}(\frac{y}{x}), P^n]$ be the maximizer of $\Gamma(y, p)$. Note that as $\frac{D(p)}{D(p)}$ is monotone by Assumption 1, $\frac{D(p)}{D(p)}$ is also monotone. In the following, we consider two cases depending on the behavior of $\frac{D(p)}{D(p)}$.

**Case 1:** $\frac{D'(p)}{D(p)}$ is increasing.

In this case, $(p - v)\frac{D'(p)}{D(p)} = p\frac{D'(p)}{D(p)} - v\frac{D'(p)}{D(p)}$ is decreasing (as $p\frac{D'(p)}{D(p)}$ is decreasing by Assumption 1). If there exists a $y_0 \in [x, +\infty)$ such that $V'(\frac{y_0}{D(p)}) + p\frac{D'(p)}{D(p)} - v\frac{D'(p)}{D(p)} > 0$ for any $p \in (D^{-1}(\frac{y_0}{x}), P^n)$, then $P(y_0) = P^n$ and $1 + P\frac{D'(p)}{D(p)} - v\frac{D'(p)}{D(p)} \geq \lim_{p \to P^n} [1 + p\frac{D'(p)}{D(p)} - v\frac{D'(p)}{D(p)}] \geq 0$ for any $p \in (D^{-1}(\frac{y_0}{x}), P^n)$. Hence, for any given $y \in (0, +\infty)$, $p = P^n$ is the maximizer of $\Gamma(y, p)$. As a result of this, we get $\Gamma(y, P^n) = (v - w)y \leq 0$, i.e., the optimal value of $\Gamma(y, p)$ is non-positive. But, this is not possible because of the following two facts: $\Gamma(0, P^n) = 0$ and $\frac{\partial \Gamma(y, p)}{\partial y} \mid_{(y, p) = (0, P^n)} = P^n - w > 0$. Therefore, for any $y \in [x, +\infty)$, there exists a unique $P(y)$ satisfying:

$$
V'(\frac{y}{D(p)}) + p\frac{D'(p)}{D(p)} - v\frac{D'(p)}{D(p)} = 0.
$$

(5)

It is obvious that $P(y)$ is unique and $P(y) > v$ for any $y \in [x, +\infty)$. Taking derivative with respect to $y$ on both sides of (5), we get:

$$
V'(\frac{y}{D(p)})\frac{y}{D(p)} + (P\frac{D'(p)}{D(p)})' - \{v\frac{D'(p)}{D(p)}\}'\frac{y}{D(p)} = 0.
$$

Hence, $P'(y) < 0$, i.e., $P(y)$ is strictly decreasing. Rewriting the above equation about $P'(y)$, we get $\frac{y}{D(P(y))} = Z(y)$ is strictly increasing, i.e., $\frac{1}{D(P(y))} = \frac{P'(y)}{D(P(y))^2} P'(y) > 0$. In summary, we can conclude that $P(y)$ is continuous and decreasing and $Z(y)$ is strictly increasing for $y \in [x, +\infty)$. It then follows that $LO(y, p, u) = \max\{y - D(P(y))u, 0\} = \max\{D(P(y))[Z(y) - u], 0\}$ is also increasing in $y$, for any realization $u$ of $\epsilon_t$. This completes the proof of part (2) of the proposition.

Next we study the behavior of $\Gamma(y, P(y))$ for $y \in [x, +\infty)$. Taking derivative of $\Gamma(y, P(y))$ with respect to $y$ we have $\frac{d\Gamma(y, P(y))}{dy} = P(y)[1 - F(Z(y))] + vF(Z(y)) - w$ and $\frac{d^2\Gamma(y, P(y))}{dy^2} = P(y)[1 - F(Z(y))] - f(Z(y))[P(y) - v]Z'(y)$. This implies that $\frac{d^2\Gamma(y, P(y))}{dy^2} \leq 0$ since $Z'(y) > 0$, i.e., $\Gamma(y, P(y))$ is strictly decreasing in $y$ for $y \in (0, +\infty)$.
\( P(y) > v \), and \( P'(y) < 0 \). Let \( \hat{Y} \) be the minimal \( y \) such that \( Z(y) = U \) (if there is no such \( y \), we define \( \hat{Y} = +\infty \)). Then \( \frac{d\Gamma(y,P(y))}{dy} = v - w < 0 \) for any \( y \in [\hat{Y}, +\infty) \) and \( \frac{d^2\Gamma(y,P(y))}{dy^2} < 0 \) for any \( y \in [x, \hat{Y}) \). Consequently, \( \Gamma(y, P(y)) \) is strictly concave for \( y \in [x, +\infty) \). Therefore, there is a unique \( y \in [x, +\infty) \) that maximizes \( \Gamma(y, P(y)) \).

If we denote the unique maximizer of \( \Gamma(y, P(y)) \) as \( S_t \), and \( s_t (\leq S_t) \) as the maximal inventory level such that \( \Gamma(s_t, P(s_t)) \leq \Gamma(S_t, P(S_t)) - K \) (if there is no such \( s_t \), let \( s_t = 0 \)), then the proof for part (1) of Proposition 1 follows.

**Case 2:** \( \frac{D'(p)}{D(p)} \) is decreasing.

Note that \( V(y, D(p)) \) is decreasing in terms of \( p \) (from Lemma 1), and for this scenario \((p - v)\frac{D'(p)}{D(p)} \) is decreasing in terms of \( p \) (based on Assumption 1). Hence, \( V(y, D(p)) + (p - v)\frac{D'(p)}{D(p)} \) is decreasing in terms of \( p \). The remainder of the proof for this case is almost identical to the case above, and so is omitted.

The optimality of the \((s_t, S_t, P_t)\) policy means that if the starting inventory level \( x \leq s_t \), then the retailer should order \((S_t - x)\) to bring the inventory level to \( S_t \) and set the optimal price to be \( P_t(S_t) \); on the other hand, if \( x > s_t \), then the retailer should not order anything and set the price as \( P_t(x) \).

### 3.2 Analysis of the Multi-Period Model

The above properties of \( \Gamma_t(y, p) \) and the definitions of \( s_t \) and \( S_t \) enable us to characterize the optimal policy for the general multi-period model. Recall that for any given initial inventory \( x \) at the beginning of period \( t \), the retailer maximizes the expected total profit given by

\[
-K_t \delta(y - x) + \Gamma_t(y, p) + \alpha \int_0^{+\infty} \Pi_{t+1}(LO_t(y, p, u)) f_t(u) du,
\]

by selecting the order-up-to inventory level \( y \) and the retail price \( p \).

We approach this optimization problem sequentially. We first determine the optimal price for a given order-up-to level \( y \), and then analyze the resulting one-variable problem in terms of \( y \). Define

\[
H_t(y) = \max_p \{\Gamma_t(y, p) + \alpha \int_0^{+\infty} \Pi_{t+1}(LO_t(y, p, u)) f_t(u) du\}. \tag{6}
\]

Let \( p_t(y) \) denote the optimal price for a given \( y \). If there are multiple optimal prices, we define \( p_t(y) \) as the smallest of them all. Let us also denote the maximizer of \( H_t(y) \) as \( y_t^* \). We now present a crucial property of \( \Gamma_t(y, p) \), which will help us to analyze \( H_t(y) \).

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Lemma 2 For any two given order-up-to levels $y^1$ and $y^2$ such that $S_t \leq y^1 < y^2$ and a given retail price $p^2$, there exists a retail price $p^1$ such that $\Gamma_t(y^1, p^1) \geq \Gamma_t(y^2, p^2)$ and $LO_t(y^1, p^1, u) \leq LO_t(y^2, p^2, u)$.

Proof: For ease of exposition, we suppress the subscript $t$ from all notation in this proof. For any given order-up-to inventory level $y \geq 0$, recall from Proposition 1 that $P(y)$ is the corresponding optimal retail price and it satisfies the following equation:

$$\{V\left(\frac{y}{D(p)}\right) + (p - v)\frac{D'(p)}{D(p)}\}_{p=P(y)} = 0.$$  

Taking derivative with respect to $y$ on both sides of the above equation, we get

$$\frac{dP(y)}{dy} = -V'\left(\frac{y}{D(P(y))}\right) \frac{1}{D(P(y))}\left(-\frac{y}{D(P(y))^2}D'(P(y))V'\left(\frac{y}{D(P(y))}\right)\right)$$

$$+ \frac{D'(P(y))}{D(P(y))} + [P(y) - v]\frac{D''(P(y))D(P(y)) - D'(P(y))^2}{D(P(y))^2}.$$  

Let $P^l(y)$ be the unique positive solution of $V\left(\frac{y}{D(p)}\right) = l$ for a given positive constant $l(>1)$ and $Z(l) = \frac{y}{D(P^l(y))}$. We also define $C = \{(y, P(y))|y > 0\}$ and $C^l = \{(y, P^l(y))|y > 0\}$. The common point on both $C$ and $C^l$, if any, is denoted by $(y^l, p^l)$. Note that $V\left(\frac{y}{D(p)}\right)$ is decreasing in terms of $p$ (by Lemma 1) and $(p - v)\frac{D'(p)}{D(p)}$ is decreasing in terms of $p$ (from the proof of Proposition 1). Hence, at $y = y^l$ we get

$$\frac{dP(y)}{dy} > -V'\left(\frac{y}{D(P(y))}\right) \frac{1}{D(P(y))}\left(-\frac{y}{D(P(y))^2}D'(P(y))V'\left(\frac{y}{D(P(y))}\right)\right)^{-1} = \frac{dP^l(y)}{dy}.$$  

We can see that both $P(y)$ and $P^l(y)$ are decreasing, but $P^l(y)$ is decreasing faster than $P(y)$ at the common point $(y^l, p^l)$. This relationship between $P(y)$ and $P^l(y)$ will play a key role in the following analysis. Note also from Proposition 1 that $S$ is the maximizer of $\Gamma(y, P(y))$ on $[0, +\infty)$.

For any given $S \leq y^1 < y^2$ and retail price $p^2$, there are two cases to consider:

Case 1 $p^2 \geq P(y^2)$: Choose $p^1 = P(y^1)$. The result then follows directly from Proposition 1.

Case 2 $p^2 < P(y^2)$: Let $l_2$ be the constant such that $(y^2, p^2)$ is on $C^{l_2}$. We need to analyze three subcases:

Subcase 2(a) There is a common point $(y^2, p^2)$ on both $C^{l_2}$ and $C$, and $y^1 \leq y^2$:

Let $p^1 = P(y^1)$. Then $p^1 \geq p^2 > P(y^2) > p^2$ and $S \leq y^1 < y^2 < y^2$. By Proposition 1, we have $\Gamma(y^1, p^1) \geq \Gamma(y^2, p^2) \geq \Gamma(y^2, p^2) \geq \Gamma(y^2, p^2)$ and $LO(y^1, p^1, u) \geq LO(y^1, p^1, u)$. 

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Thus, it is sufficient to show that $LO(y^2, p^2, u) \geq LO(y^{l_2}, p^{l_2}, u)$. For any pair $(y, p)$ on $C^{l_2}$, we define $z = \frac{y}{D(p)}$, which is a constant. Hence, the leftover $D(p)[z - u]$ of $(y, p, u)$ is decreasing in terms of $p$ on $C^{l_2}$. This proves Lemma 2 for this subcase.

**Subcase 2(b) There is a common point $(y^{l_2}, p^{l_2})$ on both $C^{l_2}$ and $C$, and $y^1 > y^{l_2}$:**

Recall that $l_2$ is the constant such that $(y^2, p^2)$ is on $C^{l_2}$. Let $(y^1, p^1)$ be the point on both $C^{l_2}$ and $\{ (y, p) | y = y^1 \}$, and $l_1$ $(< l_2)$ be the constant such that $C^{l_1}$ is passing through the point $(y^1, P(y^1))$.

For a fixed $y^i (i = 1 \text{ or } 2)$, the retail price $p^i(l)$ of the point $(y^i, p)$ on $C^{l}(l_1 \leq l \leq l_2)$ is a function of $l$ and $V\left(\frac{y^i}{D(p^i(l))}\right) = l$. Taking derivative with respect to $l$ on both sides, we get

$$\frac{dp^i(l)}{dl} = - \frac{D(p^i(l)) D(p^i(l))}{V'(y^i/D(p^i(l)))} [V'(y^i/D(p^i(l)))]^{-1}.$$ 

Next we study the behavior of $\Gamma(y^i, p^i(l))$ with respect to the curve parameter $l$. Since

$$\{ l + [p^i(l) - v] \frac{D^i(y^i(l))}{D(p^i(l))} \} = \{ D(p^i(l))(\int_0^{y^i/D(p^i(l))} u f(u) du)\}^{-1} \frac{\partial y^i}{\partial p} |_{\{p=p^i(l)\}} > 0 \text{ for any } l \geq l_1$$

(by the definition of $P(y)$) and $V' \leq 0$ (from Lemma 1), it is straightforward to see that

$$\frac{d\Gamma(y^i, p^i(l))}{dl} = - \frac{\partial \Gamma(y^i, p)}{\partial p} |_{\{p=p^i(l)\}} \frac{dp^i(l)}{dl}$$

$$\quad = - \frac{D^2(p^i(l))}{D'(p^i(l))} \left( \int_0^{y^i/D(p^i(l))} u f(u) du \right) \left( l + [p^i(l) - v] \frac{D'(p^i(l))}{D(p^i(l))} \right) \frac{1}{V'(y^i/D(p^i(l)))} \frac{D(p^i(l))}{y^i} \leq 0.$$ 

Note that based on the argument in the proof of Proposition 1, $(p - v)^{\frac{D(p)}{D(p^i(l))}}$ is always decreasing in terms of $p$. It is also true that $-\frac{D^2(p)}{D'(p)}$ is decreasing in terms of $p$, since $(\frac{D(p)}{D(p^i(l))})' = D(p)[p + \frac{D(p)}{D(p^i(l))}] > 0$ by Assumption 1. Furthermore, $\frac{y^i}{D(p^i(l))}$ is a constant for any fixed $l$. Hence, $\frac{d\Gamma(y^i, p^i(l))}{dl}$ is increasing in terms of $p^i(l)$. Let $(y^2, \tilde{p})$ be the point on $C^{l_1}$. Then

$$\Gamma(y^1, p^1) - \Gamma(y^1, P(y^1)) = \int_{l_1}^{l_2} d\Gamma(y^1, p^1(l)) \geq \int_{l_1}^{l_2} d\Gamma(y^2, p^2(l)) = \Gamma(y^2, \tilde{p}) - \Gamma(y^2, \tilde{p}).$$

On the other hand, we know that $\Gamma(y^1, P(y^1)) \geq \Gamma(y^2, P(y^2)) \geq \Gamma(y^2, \tilde{p})$ based on the unimodality of $\Gamma(y, P(y))$ and the optimality of $P(y)$. Combination of the above two inequalities yields $\Gamma(y^1, p^1) \geq \Gamma(y^2, p^2)$. Following the same argument as Subcase 2(a), we can then establish that $LO(y^2, p^2, u) \geq LO(y^1, p^1, u)$. This proves Lemma 2 for Subcase 2(b).
Subcase 2(c) No common point on $C^{l_2}$ and $C$:

Notice that for any point $(y, p)$ below $C$ we have $\frac{\partial \Gamma(y, p)}{\partial p} > 0$ (from the definition of $P(y)$ and the unimodality of $\Gamma(y, p)$ in terms of $p$). Since $(y^2, p^2)$ on $C^{l_2}$ is below $C$ for this subcase, so we always have $t_2 + (p - v)\frac{D(p)}{D'(p)} > 0$ for any $p > 0$. Also note that $t_2 > 1$; otherwise, $1 + (p - v)\frac{D(p)}{D'(p)} > 0$ for any $p > 0$, and this is not possible by the proof of Proposition 1. For any $l$ as $t_2$, the profit function $\Gamma(y, p)$ on $C^l$ can then be expressed as

$$\Gamma(p) = pD(p)[1 - \Theta(Z(l))] + v\Lambda(Z(l))D(p) - wZ(l)D(p).$$

In the following, we first study the behavior of $\Gamma(p)$. For simplicity, let $\rho = Z(l)$ in the remainder of this proof. Taking derivative of $\Gamma(p)$ with respect to $p$, we get

$$\frac{d\Gamma(p)}{dp} = D'(p)[p + D(p)\frac{D'(p)}{D'(p)}](1 - \Theta(p)) + v\Lambda(p) - w\rho.$$

As $[p + D(p)\frac{D'(p)}{D'(p)}] > 0$ (from Assumption 1) and $1 - \Theta(p) > 0$ (from Lemma 1), $\Gamma(p)$ is unimodal in terms of $p$. On the other hand, we have $\frac{d\Gamma(p)}{dp} > 0$ near $p = 0$ as $v\Lambda(p) - w\rho < -w(1 - \Theta(p)) < 0$. Therefore, the unique maximizer $p(\rho)$ of $\Gamma(p)$ is either $P^u$ or the unique solution satisfying

$$p + D(p)\frac{D'(p)}{D'(p)} = (w-v)\frac{\rho}{1-\Theta(p)} + v. \quad (7)$$

Taking derivative with respect to $p$ on both sides of $(7)$, we have $\frac{dp(\rho)}{dp} = (w-v)\{V(\rho)[1 - \Theta(p)]\}^{-1}[1 + (D')^{-1}]^{-1}$.

Notice that $\rho = Z(l)$ is decreasing in terms of $l$ and $\frac{v\Lambda(p) - w\rho}{1 - \Theta(p)} = (w-v)\frac{\rho}{1-\Theta(p)} + v$ is increasing in terms of $\rho$. Hence, if $p(\rho) = P^u$ for some $\hat{l}$, then $p(\rho) = P^u$ for any $l \geq \hat{l}$. Without loss of generality, we assume that $\hat{l}$ is the minimal one among all such $l$s for this subcase.

Now we investigate the behavior of $y(\rho)$, where $y(\rho) = \rho D(p(\rho))$. For any $l \geq \hat{l}$, we have $y(\rho) = 0$. For any $l < \hat{l}$ (there is no common point on $C$ and $C'$), the corresponding $p(\rho)$ satisfies $(7)$. Then, differentiating $y(\rho)$ with respect to $\rho$, we have $\frac{dy(\rho)}{d\rho} = D + pD'\frac{dp(\rho)}{d\rho}$.

Substitution of $\frac{dp(\rho)}{d\rho}$ in this expression yields $\frac{dy(\rho)}{d\rho} = D\{1 + (w-v)\frac{\rho}{l[1 - \Theta(p)]}D'[1 + (D')^{-1}]\}$. Using $(7)$, we then obtain

$$\frac{dy(\rho)}{d\rho} = \frac{D}{l}\{l + [(p-v)\frac{D'}{D} + 1][1 + (D')^{-1}]\}.$$

Since $l + (p-v)\frac{D'}{D} > 0$ for any point $(y, p)$ on $C^l$ (recall that there is no common point on both $C'$ and $C$ for this subcase), for establishing that $y(\rho)$ is increasing in $\rho$, it is sufficient
to show that \([p-v\frac{D'}{P} + 1][1+\frac{(D')^2}{D^2}]^{-1} \geq (p-v)\frac{D'}{P}\), which simplifies to \(\frac{D'}{P}(\frac{D'}{P})' \leq \frac{1}{p-v}\). As \(p > v \geq 0\) on \(C^l\) by (7), it is then sufficient to prove that \(\frac{D'}{P}(\frac{D'}{P})' \leq \frac{1}{p}\). This is equivalent to having a decreasing \(\frac{D'}{P}\), which itself is assured by Assumption 1. Hence \(y(\rho)\) is increasing in terms of \(\rho\), i.e., it is decreasing in terms of \(l\). Based on the proof for subcase 2(b), we know that for any \(C^l\)-type curve as defined in that subcase, the corresponding maximizer \((\hat{y}, \hat{p})\) of \(\Gamma(y, p)\) on that curve should satisfy \(\hat{y} \leq S\). Combining this and the decreasing property of \(y(\rho)\) in terms of \(l\), we can say that \(y(\rho) \leq S\) for any \(l\) such that there is no any common point on both \(C\) and \(C^l\) (where \(\rho = Z(l)\) and \(V(Z(l)) = l\)).

Let \(p^1 = P^{l_2}(y^1)\). Based on the unimodality of \(\Gamma(p)\) and \(y(Z(l_2)) \leq S\), we then have \(\Gamma(y^1, p^1) \geq \Gamma(y^2, p^2)\). The relation \(L_0(y^1, p^1, u) \leq L_0(y^2, p^2, u)\) follows using the same argument as subcases 2(a) and 2(b). So Lemma 2 is also valid for subcase 2(c).

The above proofs for Case 1 and Case 2 establishes Lemma 2. \(\square\)

The result in Lemma 2 is slightly stronger and hence implies “Condition 1” of H&J for multiplicative demand with lost sales.\(^7\) In non-technical terms, it means that: i) the nearer the inventory level at the beginning of period \(t\) to \(S_t\), the better it is for the retailer from the viewpoint of expected profit; and ii) if the retailer starts with a higher inventory level at the beginning of the period, this will result in higher leftovers at the end of the period.

Lemma 2 helps us in characterizing the optimal optimal replenishment strategy when the initial stock level is “high” (i.e., \(x \geq S_t\)). But before doing so, we make the following assumption about the nature of non-stationarity in our model:

**Assumption 3** Suppose that \(K_l \geq \alpha K_{l+1}\) and \(S_l \leq S_{l+1}\) for any \(l\) \((t \leq l < T)\).

Note that the above assumption is prevalent in the related literature (e.g., H&J; Chen and Simchi-Levi 2004a). For example, we would expect the fixed ordering cost to diminish over time as a result of experience and learning. Similarly, \(S_t\) would be an increasing sequence in settings with constant model parameters over the selling horizon or when \(\{\epsilon_t\}\) is a sequence of increasing random variables (refer to H&J). We now present the following:

\(^7\)“Condition 1” of H&J can be stated in our context as: \(\Gamma_t(y) = \max_p \Gamma_t(y, p)\) is quasi-concave, and for any \(y^1\) and \(y^2\) satisfying \(S_t \leq y^1 < y^2\) and \(p^2\), there exists a \(p^1\) such that \(\Gamma_t(y^1, p^1) \geq \Gamma_t(y^2, p^2)\) and \(L_0(y^1, p^1, u) \leq \max\{S_t, L_0(y^2, p^2, u)\}\).
Lemma 3 Suppose that Assumption 3 is valid. Then it is optimal for the retailer to order nothing at the beginning of period \( t \) if the initial stock level \( x \geq S_t \).

Proof: Follows from Lemma 2 and Proposition 3.2 in H&J (this result is similar to Corollary 3.3 of H&J).

In order to characterize the optimal policy when the initial stock level is lower (i.e., \( x \in [0, S_t) \)), we require the following result.

Lemma 4 Let \( \rho_t = \min_{t \leq l \leq T} s_l \), and suppose that Assumption 3 is valid. Then the following are true:

1. \( H_t(y) \) is strictly increasing and \( H_t(y) \leq H_t(S_t) - K_t \) for any \( y \in [0, \rho_t] \).
2. \( H_t(y) > H_t(\rho_t) \) and \( H_t(y) \leq K_t + H_t(\bar{y}) \) for any \( \rho_t < y \leq \bar{y} \leq S_t \).

Proof: By Proposition 1, the lemma holds true for \( t = T \). Now suppose that it holds true for period \( t + 1 \) (\( \leq T \)). In the following we will show it also holds true for period \( t \) (from induction hypothesis the lemma then holds for all \( t \)).

For period \( t + 1 \), we have \( H_{t+1}(y) \leq H_{t+1}(S_{t+1}) - K_{t+1} \) and \( H_{t+1}(y) \) is strictly increasing on \([0, \rho_{t+1}]\). This means that for any \( x \in [0, \rho_{t+1}] \), it is always optimal to order up to \( y^*_{t+1} \) at the beginning of period \( t + 1 \), and \( \Pi_{t+1}(x) \) is constant (say, \( V_{t+1} \)) on \([0, \rho_{t+1}]\). Then, for any \( y \in [0, \rho_t] \), \( H_t(y) \) given by (6) can be written as \( H_t(y) = \Gamma_t(y, P_t(y)) + \alpha V_{t+1} \). Thus, \( H_t(y) \) is strictly increasing on \([0, \rho_t]\), which proves the first result in part (1).

Suppose that \( y^*_{t+1} \geq S_{t+1} \). Then it is obvious that for any \( \rho_t < y \leq \bar{y} \leq S_{t+1} \),

\[
\Pi_{t+1}(y) \geq \Pi_{t+1}(\rho_{t+1}) = \Pi_{t+1}(\rho_t) \quad \text{and} \quad |\Pi_{t+1}(y) - \Pi_{t+1}(\bar{y})| \leq K_t.
\]

(8)

Suppose now that \( y^*_{t+1} \in (\rho_{t+1}, S_{t+1}) \). If \( y \in (\rho_t, y^*_{t+1}] \), clearly \( \Pi_{t+1}(y) \geq \Pi_{t+1}(\rho_{t+1}) \). If \( y \in (y^*_{t+1}, S_{t+1}] \), as \( H_{t+1}(y) \geq H_{t+1}(y^*_{t+1}) - K_{t+1} = \Pi_{t+1}(\rho_{t+1}) \) from the induction hypothesis, we get \( \Pi_{t+1}(y) \geq \Pi_{t+1}(\rho_{t+1}) \). Thus, the first inequality in (8) holds true in this case too. Similarly, we can argue that the second inequality in (8) also holds true if \( y^*_{t+1} \in (\rho_{t+1}, S_{t+1}) \). In order to show this, we consider the following three cases:

- If both \( y \) and \( \bar{y} \) are in \((\rho_t, y^*_{t+1}]\), then the second inequality in (8) is clearly true.
• If \( y \in (\rho_t, y^*_t+1] \) and \( \bar{y} > y^*_t+1 \), we claim that \( \Pi_{t+1}(\bar{y}) = H_{t+1}(\bar{y}) \). Otherwise, there exists a \( y_2 > \bar{y} \) such that \( H_{t+1}(\bar{y}) < H_{t+1}(y_2) - K_{t+1} \leq H_{t+1}(y^*_t+1) - K_{t+1} \). As \( y^*_t+1 < \bar{y} < S_{t+1} \), this is a contradiction from the induction hypothesis. Hence, we get \( \Pi_{t+1}(y) = H_{t+1}(y^*_t+1) - K_{t+1} \) \( \geq H_{t+1}(\bar{y}) - K_{t+1} = \Pi_{t+1}(\bar{y}) - K_{t+1} \) and \( \Pi_{t+1}(y) \leq H_{t+1}(y^*_t+1) \leq K_{t+1} + H_{t+1}(\bar{y}) = K_{t+1} + \Pi_{t+1}(\bar{y}) \). This proves the second inequality in (8).

• If \( y \in (y^*_t+1, S_{t+1}] \), then both \( \Pi_{t+1}(\bar{y}) \) and \( \Pi_{t+1}(y) \) are in \( [\Pi_{t+1}(y^*_t+1) - K_{t+1}, \Pi_{t+1}(y^*_t+1)] \) from the induction hypothesis. Again the second inequality in (8) holds true.

Thus, both inequalities in (8) hold true for any \( \rho_t < y \leq \bar{y} \leq S_t \).

Recall that \( \Pi_{t+1}(y) = V_{t+1} \), a constant for any \( y \in [0, \rho_t] \). Define \( \dot{H}_t(y) = \Gamma_t(y, P_t(y)) + \alpha \int_0^{\infty} \Pi_{t+1} \left( LO_t(y, P_t(y), u) \right) f_t(u) du \). As \( \Pi_{t+1}(y) \geq \Pi_{t+1}(\rho_t) = V_{t+1} \) for any \( y \in (\rho_t, S_t] \) by the first inequality in (8), we get

\[
H_t(S_t) \geq \dot{H}_t(S_t) \geq \Gamma_t(S_t, P_t(S_t)) + \alpha V_{t+1} \geq \Gamma_t(\rho_t, P_t(\rho_t)) + K_t + \alpha V_{t+1}.
\]

The last inequality is obtained from the definition of \( s_t \) and the relation \( \rho_t \leq s_t \). Hence, \( H_t(S_t) \geq \dot{H}_t(\rho_t) + K_t = H_t(\rho_t) + K_t \). Thus, \( H_t(y) \leq H_t(\rho_t) \leq H_t(S_t) - K_t \) for any \( y \in [0, \rho_t] \). This completes the proof of part (1) of the lemma.

As regards the proof of part (2) of the lemma, for any \( y \in (\rho_t, S_t] \), the first inequality in (8) implies that

\[
H_t(y) \geq \dot{H}_t(y) \geq \Gamma_t(y, P_t(y)) + \alpha V_{t+1} \geq \Gamma_t(\rho_t, P_t(\rho_t)) + \alpha V_{t+1} = \dot{H}_t(\rho_t) = H_t(\rho_t),
\]

which establishes the first result in part (2). For the second result in part (2), observe that for any given \( \rho_t < y \leq \bar{y} \leq S_t \),

\[
H_t(y) = \Gamma_t(y, P_t(y)) + \alpha \int_0^{\infty} \Pi_{t+1}(LO_t(y, p_t(y), u)) f_t(u) du,
\]

\[
\leq \Gamma_t(y, P_t(y)) + \alpha \int_0^{\infty} \Pi_{t+1}(LO_t(y, p_t(y), u)) f_t(u) du,
\]

\[
\leq \Gamma_t(\bar{y}, P_t(\bar{y})) + \alpha \int_0^{\infty} \Pi_{t+1}(LO_t(y, p_t(y), u)) f_t(u) du,
\]

\[
= \dot{H}_t(\bar{y}) + \alpha \int_0^{\infty} [\Pi_{t+1}(LO_t(y, p_t(y), u)) - \Pi_{t+1}(LO_t(\bar{y}, P_t(\bar{y}), u))] f_t(u) du.
\]

Since \( \Pi_{t+1}(y) \) is a constant for \( y \in [0, \rho_t] \) and \( \Pi_{t+1}(y) \) is continuous at \( \rho_t \), the above inequality and the second inequality in (8) together implies that \( H_t(y) \leq \dot{H}_t(\bar{y}) + \alpha K_{t+1} \leq H_t(\bar{y}) + K_t \). □

Lemmas 3 and 4 jointly characterize the optimal dynamic policy for the multi-period model.
**Theorem 1** Suppose that Assumption 3 is valid. Then, at the beginning of period \( t \), if the initial stock level \( x \leq \rho_t \) or if \( x \in (\rho_t, S_t) \) and it is optimal to order, then the retailer should order-up-to \( y^*_t \) and set the price as \( p_t(y^*_t) \). For any other initial stock level \( x \), the retailer’s optimal policy is to order nothing and charge the price \( p_t(x) \).

**Proof:** We only need to show that for any initial stock level \( x \), if it is optimal to order, then the optimal order-up-to inventory level is \( y^*_t \). We prove this by a contradiction argument. Suppose that the above claim is not true. Then there should exist \( x_1 < x_1^* < x_2 < x_2^* \) such that at \( x_1 \) it is optimal to order up to \( x_1^* \) and at \( x_2 \) it is optimal to order up to \( x_2^* \). Then we must have \( x_1 < x_2 \leq S_t \) by Lemma 3, and \( H_t(x_1^*) > K_t + H_t(x_2) \). But this is not possible based on Lemma 4. This proves the theorem. \( \square \)

**Remark:** Note that for Lemmas 2 and 4, and consequently Theorem 1, to hold true, we only require Lemma 1, Proposition 1 (in each period) and Assumption 3 to be valid.

Theorem 1 clearly establishes the optimal replenishment policy when the starting inventory level is either low (\( x \leq \rho_t \)) or high (\( x \geq S_t \)). The complication arises in the intermediate range (\( \rho_t < x < S_t \)), where it is not possible to ascertain the exact behavior of the profit function \( H_t(y) \). Polatoglu and Sahin (2000) indicate that multiple order-up-to levels might exist (for general \( D_t(p) \) functions). Nevertheless, we are able to show that for a sufficiently large group of demand functions, there is a unique order-up-to level, whenever it is optimal to order. We are also able to specify the optimal price to charge in each period based on the post-replenishment inventory level. A careful examination of Theorem 1 reveals that although we use different notation, the structure of the optimal policy is of the form \((s, S, A, p)\), as shown to be optimal for the backordering case by Chen and Simchi-Levi (2004a).\(^8\) If we denote the set of inventory levels \( \in (\rho_t, S_t) \) for which it is optimal to order as \( A_t \), our definition of \((\rho_t, y^*_t, S_t, A_t)\) is equivalent to Chen and Simchi-Levi’s \((s_t, S_t, (s_t + S_t)/2, A_t)\), respectively (see §3). We can also generate a simple upper bound on the optimal order-up-to level \( y^*_t \), utilizing our characterization of \( H_t(y) \) in Lemma 4, which is useful for computational purposes.

**Proposition 2** Let \( m_t \ (\geq S_t) \) be the maximal \( y \) such that \( \Gamma_t(y, P_t(y)) \geq \Gamma_t(S_t, P_t(S_t)) - K_t \). Then, \( y^*_t \in [\rho_t, m_t] \).

\(^8\)This is also true for stationary, finite horizon problems in our framework.
Proof: For any $y \in [0, \rho_t)$, we have $H_t(y) < H_t(\rho_t) \leq H_t(y^*_t)$ by Lemma 4. For any $y \in (m_t, +\infty)$,

$$H_t(y) = \Gamma_t(y, p_t(y)) + \alpha \int_0^{+\infty} \Pi_{t+1}(LO_t(y, p_t(y), u)) f_t(u) du$$

$$\leq \Gamma_t(y, P_t(y)) + \alpha \int_0^{+\infty} \Pi_{t+1}(LO_t(y, p_t(y), u)) f_t(u) du$$

$$< \Gamma_t(S_t, P_t(S_t)) - K_t + \alpha \int_0^{+\infty} \Pi_{t+1}(LO_t(y, p_t(y), u)) f_t(u) du$$

$$= \hat{H}_t(S_t) - K_t + \alpha \int_0^{+\infty} [\Pi_{t+1}(LO_t(y, p_t(y), u)) - \Pi_{t+1}(LO_t(S_t, P_t(S_t), u))] f_t(u) du,$$

where $\hat{H}_t(y) = \Gamma_t(y, P_t(y)) + \alpha \int_0^{+\infty} \Pi_{t+1}(LO_t(y, P_t(y), u)) f_t(u) du$ as defined in the proof of Lemma 4. If $LO_t(y, p_t(y), u) \leq S_{t+1}$, then $\Pi_{t+1}(LO_t(y, p_t(y), u)) - \Pi_{t+1}(LO_t(S_t, P_t(S_t), u)) \leq K_{t+1}$ by (8) in the proof of Lemma 4. If, on the other hand, $LO_t(y, p_t(y), u) > S_{t+1}$, then

$$\Pi_{t+1}(LO_t(y, p_t(y), u)) = H_{t+1}(LO_t(y, p_t(y), u)) \leq H_{t+1}(y^*_{t+1}).$$

In this case, we consider two scenarios:

- If $y^*_{t+1} \leq S_{t+1}$, then by the second inequality of (8) in the proof of Lemma 4 we get

$$\Pi_{t+1}(LO_t(y, p_t(y), u)) - \Pi_{t+1}(LO_t(S_t, P_t(S_t), u)) \leq H_{t+1}(y^*_{t+1}) - H_{t+1}(LO_t(S_t, P_t(S_t), u)) \leq K_{t+1}.$$

- If $y^*_{t+1} > S_{t+1}$, then

$$\Pi_{t+1}(LO_t(y, p_t(y), u)) - \Pi_{t+1}(LO_t(S_t, P_t(S_t), u)) \leq H_{t+1}(y^*_{t+1}) - \Pi_{t+1}(LO_t(S_t, P_t(S_t), u)) \leq K_{t+1}.$$

Thus, for any $y \in (m_t, +\infty)$, we obtain $H_t(y) < \hat{H}_t(S_t) - K_t + \alpha K_{t+1} \leq H_t(S_t) \leq H_t(y^*_t)$. Since $H_t(y) < H_t(y^*_t)$ for both $y \in [0, \rho_t)$ and $y \in (m_t, +\infty)$, $y^*_t \in [\rho_t, m_t]$.

Furthermore, if $K_t = 0$ for any $t$ ($t \leq l < T$), then from the proof of Lemma 4 it is clear that $\rho_t = s_t = S_t$ and the set $A_t$ disappears, implying that a base-stock policy is optimal at the beginning of period $t$. Likewise, we can also characterize the optimal dynamic decision policy for a stationary infinite planning horizon scenario.

Theorem 2 For the stationary infinite horizon problem, an $(s, S, p)$ policy is optimal.

Proof: Follows from Lemma 2 and Theorem 3.4 in H&J. □

Note that all the results for the lost sales case continues to hold even if excess demands in each period are backordered at a cost of $b_t(\geq 0)$ per unit, where $0 \leq h_t < b_t$ for any $t(1 \leq t \leq T)$. In fact, we then only require the deterministic demand function $D_t(p)$ to satisfy the following assumption (for any $t, 1 \leq t \leq T$):
Assumption 4 The demand function $D_t(p)$ is positive, strictly decreasing, and continuously differentiable in terms of $p$ on $(0, P_t^u)$ and $\lim_{p \to P_t^u} p D_t(p) = 0$. Furthermore, $p + \frac{D_t(p)}{D_t'(p)}$ is strictly increasing for $p \in (0, P_t^u)$. 

The strictly increasing property of $p + \frac{D_t(p)}{D_t'(p)}$ for $p \in (0, P_t^u)$ in the above assumption is exactly equivalent to Assumption 2 in Chen and Simch-Levi (2004a, page 888).

Obviously, the backordering scenario will result in some changes in the analytical expressions compared to the lost sales case (e.g., $g_t(y, p)$, and consequently, $\pi_t(x)$). We do not repeat the detailed derivations and proofs here. However, we can show that Proposition 1 still holds true, and based on that we can then prove that so do Lemmas 2 and 4. Hence, Theorem 1 remains valid for that scenario. That is, we can re-establish Chen and Simchi-Levi’s (2004a) result that the optimal policy for the finite horizon, backordering model is also of the $(s, S, A, p)$ form, if the inventory holding and backordering costs are linear. This immediately leads to results analogous to zero fixed cost and stationary infinite horizon cases like before.

4 Conclusions

In this paper, we studied the optimal dynamic joint pricing and inventory control problem of a retailer facing price-sensitive stochastic demand. Other salient features of our framework include positive fixed ordering costs, lost sales, multiplicative demand form, and non-stationary demand and cost parameters (although the parameters need to satisfy certain mild restrictions).

Our primary contribution is the proof of optimality of a $(s, S, A, p)$ policy for a finite selling horizon problem (which reduces to a $(s, S, p)$ policy in a stationary, infinite horizon setting). We established this result for a general class of demand functions. Considering that the form of the optimal policy was not known for the multiplicative demand form with lost sales, our work addresses an important theoretical gap and complements recent results by Chen and Simchi-Levi (2004 a,b) and H&J.

From a technical point of view, we derived a key property, which implies “Condition 1” of H&J, to prove the no-ordering property for high initial stock levels. To complete the characterization of the optimal inventory policy, we derived additional properties and showed the uniqueness

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9 Chen and Simch-Levi (2004a) assume that $D_t p(D_t)$ is concave in terms of $D_t$ where $p(D_t)$ is the inverse function of $D_t(p)$.

10 Details are available from the authors on request.
of the optimal order-up-to levels for all initial stock levels with positive ordering quantities. This latter result is a novel contribution to the literature. As regards to the pricing policy, we only specify the optimal price to charge in each period, based on the post-replenishment inventory level. Note that the technique for characterizing the optimal policy can be used in other dynamic inventory control problems involving joint pricing (and possibly other) decisions. Particularly, the strength of our technique lies in the fact that it readily applies to any scenario in which the single-period profit function behaves as indicated in Proposition 1.\footnote{Our requirement on the behavior is less restrictive compared to “Condition 2” of H&J. Specifically, H&J specify certain behavior for $\Gamma_t(y, p)$ on the $(p, y)$ plane for optimality of the $(s, S, p)$ policy, while our requirements are for $\Gamma_t(y, P_t(y))$ on $P_t(y)$ curve for optimality of the $(s, S, A, p)$ policy.} For example, using this technique we can re-establish Chen and Simchi-Levi’s (2004a) result regarding the optimality of the $(s, S, A, p)$ policy for the scenario in which excess demands are backordered at a per unit backordering cost. Based on recent results of Huh and Janakiraman (2004), our optimal policy for the infinite horizon case will remain valid even when the retailer maximizes (infinite horizon) average cost as opposed to total discounted cost. Lastly, we would like to point out that our technique enables constructive optimal policy results like H&J using DP approach; however, we do not make use of concepts like $K − convexity$ normally associated with DP in the related literature.

References


